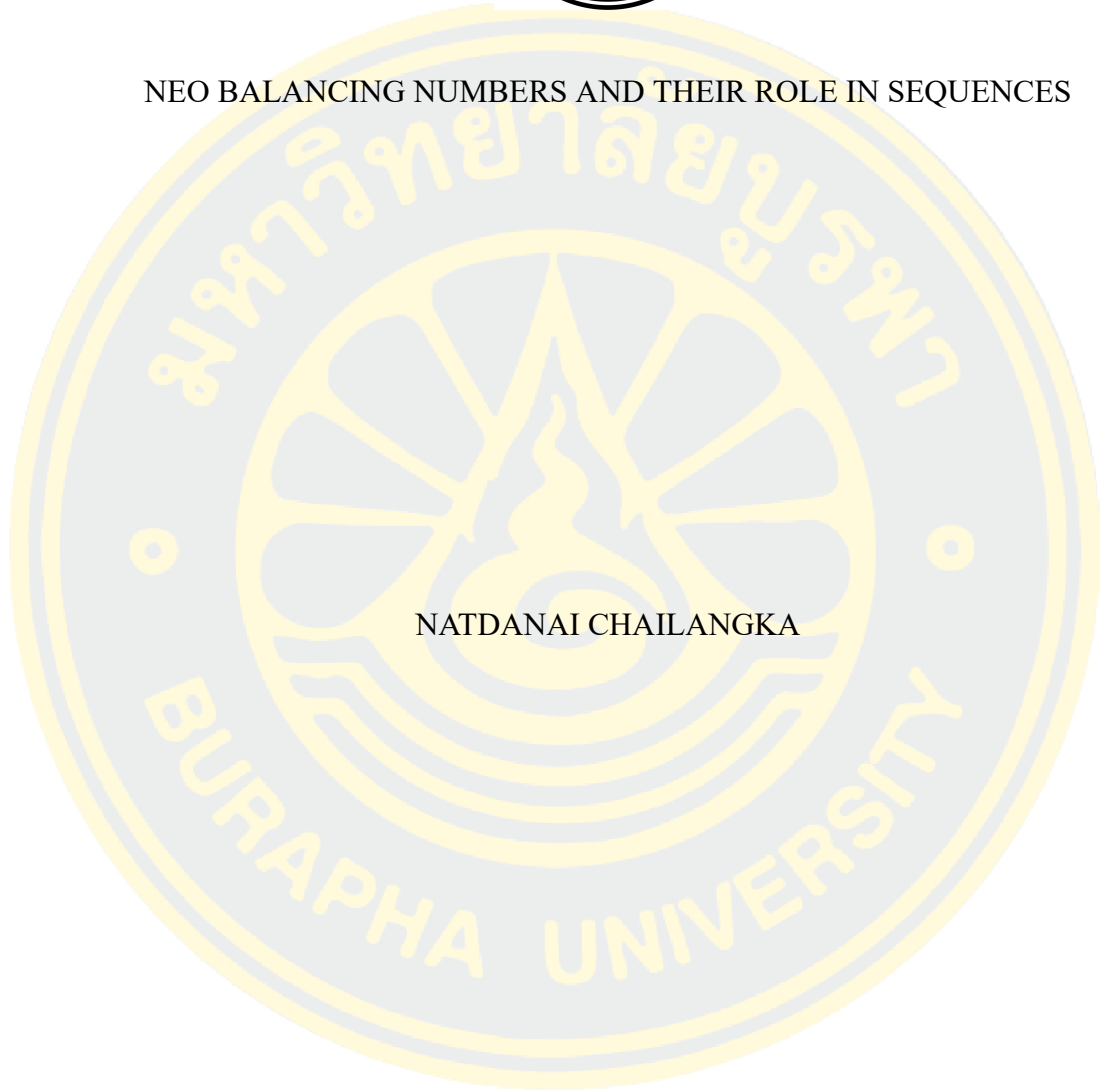




NEO BALANCING NUMBERS AND THEIR ROLE IN SEQUENCES



NATDANAI CHAILANGKA

A DISSERTATION PROPOSAL SUBMITTED IN PARTIAL FULFILLMENT OF  
THE REQUIREMENTS FOR DOCTOR DEGREE OF PHILOSOPHY

IN MATHEMATICS

FACULTY OF SCIENCE

BURAPHA UNIVERSITY

2025

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The Dissertation of Natdanai Chailangka has been approved by the examining committee to be partial fulfillment of the requirements for the Doctor Degree of Philosophy in Mathematics of Burapha University.

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This dissertation focuses on the study of neo balancing numbers and their relationships to classical balancing numbers,  $t$ -balancing numbers, and sequence  $t$ -balancing numbers. We introduce the concept of  $t$ -neo balancing numbers and derive their fundamental properties, including generating functions and recurrence relations. Furthermore, we explore the generalization of neo balancing numbers within sequences defined by linear expressions, and establish several significant results linking neo balancing numbers to Diophantine equations and Pell-type structures via important Brahmagupta's identity. Thus, we discover generalized sequence neo balancing numbers and their theorem. Furthermore, we also study certain sequences on sequence  $t$ -neo balancing number which can be expanded in the future. The findings contribute to the deeper understanding in number theory and open new directions for future research.

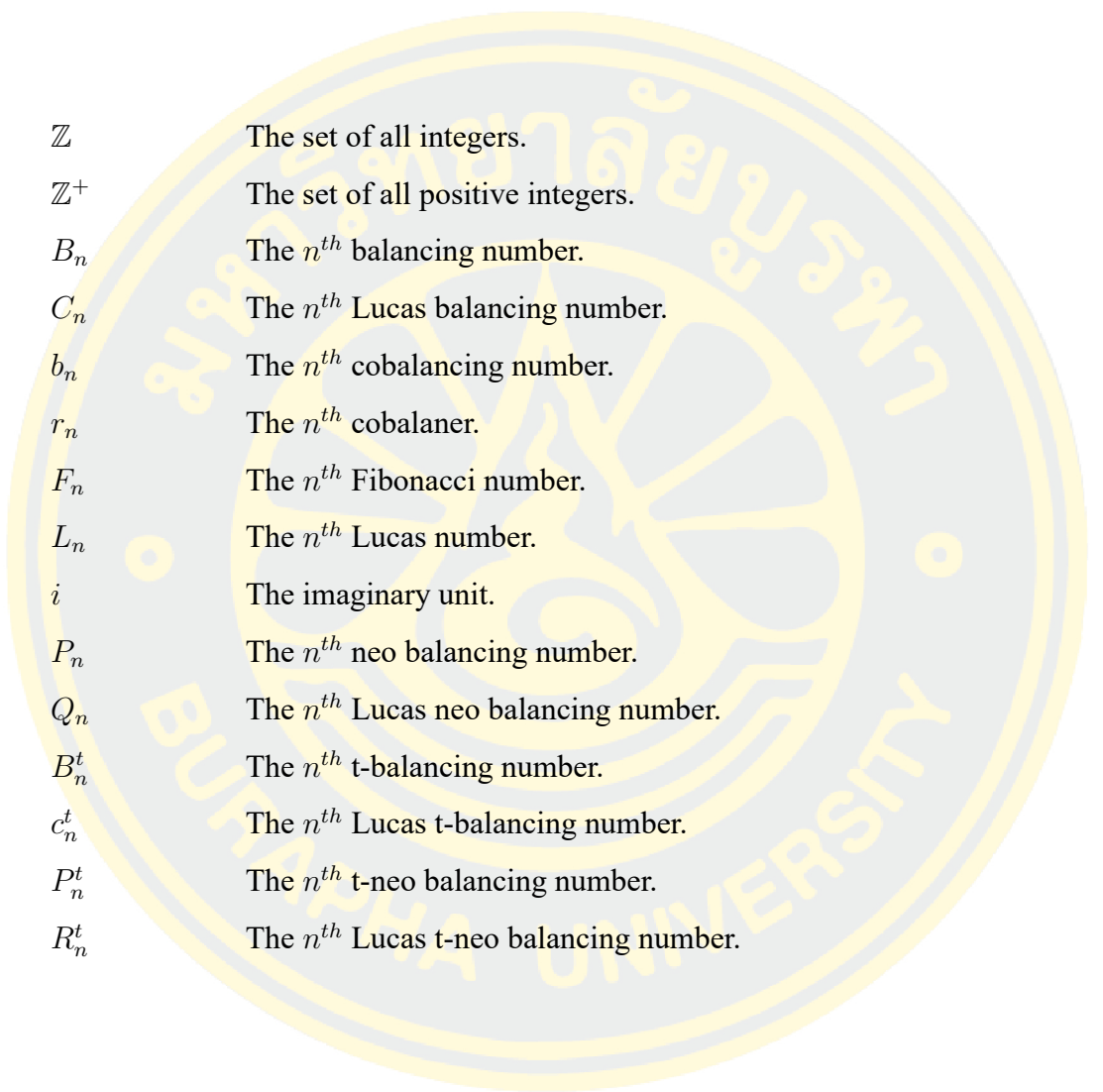
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## Notation



$\mathbb{Z}$	The set of all integers.
$\mathbb{Z}^+$	The set of all positive integers.
$B_n$	The $n^{\text{th}}$ balancing number.
$C_n$	The $n^{\text{th}}$ Lucas balancing number.
$b_n$	The $n^{\text{th}}$ cobalancing number.
$r_n$	The $n^{\text{th}}$ cobalancer.
$F_n$	The $n^{\text{th}}$ Fibonacci number.
$L_n$	The $n^{\text{th}}$ Lucas number.
$i$	The imaginary unit.
$P_n$	The $n^{\text{th}}$ neo balancing number.
$Q_n$	The $n^{\text{th}}$ Lucas neo balancing number.
$B_n^t$	The $n^{\text{th}}$ t-balancing number.
$c_n^t$	The $n^{\text{th}}$ Lucas t-balancing number.
$P_n^t$	The $n^{\text{th}}$ t-neo balancing number.
$R_n^t$	The $n^{\text{th}}$ Lucas t-neo balancing number.

# CHAPTER 1

## INTRODUCTION

In this chapter, we introduce some background on balancing numbers and neo balancing numbers. Furthermore, we present some interesting results on balancing numbers and neo balancing numbers.

### Introduction

Behera and Panda (1999) discovered balancing numbers via studying the square roots of triangular numbers. Then the balancing numbers and fascinating properties of balancing numbers were found and many researches about balancing numbers have been extensively investigated and generalized in various ways. Over the years, Panda (2007) studied interesting sequences in balancing numbers and some properties. Later on Dash and Ota (2012) succeeded in generalizing balancing numbers in an innovative way and introduced  $t$ -Balancing numbers. Chailangka and Pakapongpun (2021) defined and introduced neo balancing numbers and then investigated a generating function, recurrence relations and an application to a Diophantine equation of neo balancing numbers. Moreover, the relations between neo balancing number and balancing number and the connection of analogous properties of neo balancing numbers were also provided.

Motivated by prior studies of balancing numbers, we initiated researching and questioning on properties and generalization of neo balancing numbers. We expect that neo balancing numbers can be generalized and we investigate some fascinating properties of neo balancing numbers.

### Research Objectives

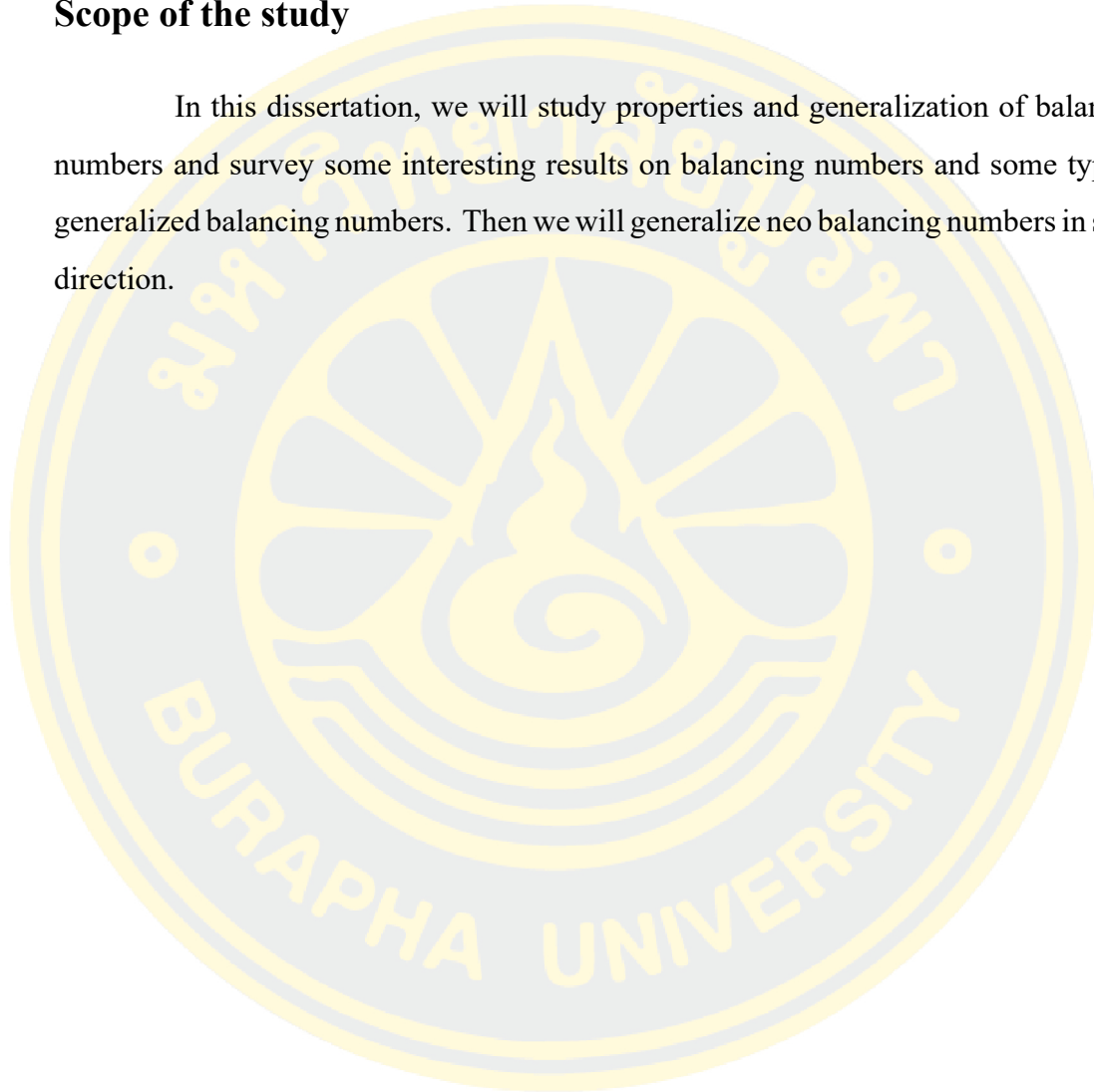
1. To study properties and generalization of balancing numbers.
2. To survey some interesting results on balancing numbers and some type of

generalized balancing numbers.

3. To generalize neo balancing numbers in some direction.

### **Scope of the study**

In this dissertation, we will study properties and generalization of balancing numbers and survey some interesting results on balancing numbers and some type of generalized balancing numbers. Then we will generalize neo balancing numbers in some direction.



## CHAPTER 2

### PRELIMINARIES AND LITERATURE REVIEWS

In this chapter, we will present important preliminary notes and some literature reviews concerning balancing numbers, neo balancing numbers, sequence balancing numbers, sequence cobalancing numbers, t-balancing numbers and sequence t-balancing numbers.

#### Preliminaries

##### Triangular numbers

**Definition 2.1.** (Pappas, 1989) *The triangular number  $T_n$  is a figurate number that can be represented in the form of a triangular grid of points where the first row contains a single element and each subsequent row contains one more element than the previous one. For example  $T_1 = 1, T_2 = 3, \dots$ . The triangular numbers are therefore  $1, 1+2, 1+2+3, 1+2+3+4, \dots$ , so for  $n = 1, 2, \dots$ , the first few are  $1, 3, 6, 10, 15, 21, \dots$*

##### Diophantine equation

**Definition 2.2.** (Sundstrom, 2006) *An equation whose solutions are required to be integers is called a Diophantine equation.*

##### Pell's equation

**Definition 2.3.** (Rosen, 1986) *Pell's equation, also called the Pell–Fermat equation, is any Diophantine equation of the form*

$$x^2 - ny^2 = 1,$$

*where  $n$  is a given positive non square integer, and integer solutions are sought for  $x$  and  $y$ .*

### Additional solutions from the fundamental solution for Pell's equation

**Lemma 2.4.** (Rosen, 1986) *Let  $k$  be a non negative integer. If  $(x_1, y_1)$  is the fundamental solution of Pell's equation, then all remaining solutions can be calculated by algebraically from*

$$x_k + y_k\sqrt{n} = (x_1 + y_1\sqrt{n})^k.$$

Moreover, the remaining solution also can be calculated by recurrence relations

$$x_{k+1} = x_1x_k + ny_1y_k$$

$$y_{k+1} = x_1y_k + y_1x_k,$$

where  $n$  is a given positive non square integer, and integer solutions are sought for  $x$  and  $y$ .

### Fibonacci sequence

**Definition 2.5.** (Vajda, 2008) *The first of two Fibonacci numbers are  $F_1 = 1, F_2 = 1$ , and other terms of the sequence are obtained by means of the recurrence relation*

$$F_{n+1} = F_n + F_{n-1}, n \geq 2.$$

### Lucas sequence

**Definition 2.6.** (Vajda, 2008) *Lucas Sequence is also obtained from the same recurrence relation as that for Fibonacci numbers. The first two Lucas numbers are  $L_1 = 1, L_2 = 3$  and other terms of the sequence are obtained by means of the recurrence relation*

$$L_{n+1} = L_n + L_{n-1}, n \geq 2.$$

### Cobalancing numbers

**Definition 2.7.** (Panda & Ray, 2005) *An integer  $n \in \mathbb{Z}^+$  is called a cobalancing number if  $n$  is a solution of*

$$1 + 2 + 3 + \cdots + n = (n + 1) + (n + 2) + \cdots + (n + r) \quad (2.1)$$

for some  $r \in \mathbb{Z}^+$ . Here  $r$  is called the cobalancer corresponding to the balancing number  $n$ . For example, 2, 14, 84 are balancing numbers with balancers 1, 6, 35, respectively.

### Brahmagupta's identity

**Theorem 2.8.** (Shenitzer & Stillwell, 2002) For  $a, b, c, d, N \in \mathbb{C}$ ,

$$(a^2 + Nb^2)(c^2 + Nd^2) = (ac \pm Nbd)^2 + N(ad \mp bc)^2.$$

**Theorem 2.9.** (Dutta, 2003) If the triples  $(x_1, y_1, k_1)$  and  $(x_2, y_2, k_2)$  are solution of

$$x^2 - ny^2 = k,$$

then

$$(x_1x_2 + ny_1y_2, x_1y_2 + x_2y_1, k_1k_2) \text{ and } (x_1x_2 - ny_1y_2, x_1y_2 - x_2y_1, k_1k_2)$$

are also solutions of the above equation.

## Relevant research

### Balancing numbers

First of all, Behera and Panda published an important result on the square roots of triangular numbers and introduced the balancing numbers and many properties of these numbers.

**Definition 2.10.** (Behera & Panda, 1999) An integer  $n \in \mathbb{Z}^+$  is called a balancing number if  $n$  is a solution of

$$1 + 2 + 3 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r) \quad (2.2)$$

for some  $r \in \mathbb{Z}^+$ . Here  $r$  is called the balancer corresponding to the balancing number  $n$ . For example, 6, 35, 204 are balancing numbers with balancers 2, 14, 84, respectively.

We can rewrite equation (2.2) to

$$n^2 = \frac{(n + r)(n + r + 1)}{2} \quad (2.3)$$

or

$$r = \frac{-(2n+1) + \sqrt{8n^2+1}}{2}. \quad (2.4)$$

From (2.3) and (2.4), we obtain valuable results for the balancing number  $n$ .

**Theorem 2.11.** (Behera & Panda, 1999) For  $n \in \mathbb{Z}^+$ ,  $n$  is a balancing number if and only if  $n^2$  is a triangular number.

**Theorem 2.12.** (Behera & Panda, 1999) For  $n \in \mathbb{Z}^+$ ,  $n$  is a balancing number if and only if  $8n^2 + 1$  is a perfect square.

Furthermore, they also obtained many fascinating properties for balancing numbers.

**Lemma 2.13.** (Behera & Panda, 1999) If  $x$  is an even balancing number, then  $3x + \sqrt{8x^2 + 1}$  is odd balancing number.

**Lemma 2.14.** (Behera & Panda, 1999) If  $x$  is an odd balancing number, then  $3x + \sqrt{8x^2 + 1}$  is even balancing number.

**Lemma 2.15.** (Behera & Panda, 1999) If  $B_n$  is the  $n^{\text{th}}$  balancing number, then  $B_{n+1} = 3B_n + \sqrt{8B_n^2 + 1}$  and  $B_{n-1} = 3B_n - \sqrt{8B_n^2 + 1}$ .

**Lemma 2.16.** (Behera & Panda, 1999) If  $B_n$  is the  $n^{\text{th}}$  balancing number, then its recurrence relation is  $B_{n+1} = 6B_n - B_{n-1}$  when  $B_0 = 1$  and  $B_1 = 6$ .

**Theorem 2.17.** (Behera & Panda, 1999) Let  $B_n$  be the  $n^{\text{th}}$  balancing number. Then

$$(a) B_{n+1} \cdot B_{n-1} = (B_n + 1)(B_n - 1).$$

$$(b) B_n = B_k \cdot B_{n-1} - B_{k-1} \cdot B_{n-k-1} \text{ for any positive integer } k < n.$$

$$(c) B_{2n} = B_n^2 - B_{n-1}^2.$$

$$(d) B_{2n+1} = B_n(B_{n+1} - B_{n-1}).$$

**Lemma 2.18.** (Behera & Panda, 1999) If  $B_n$  is the  $n^{\text{th}}$  balancing number, then its Binet form is

$$B_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}; n = 0, 1, 2, \dots, \quad (2.5)$$

where  $\lambda_1 = 3 + \sqrt{8}$  and  $\lambda_2 = 3 - \sqrt{8}$ .

**Theorem 2.19.** (Behera & Panda, 1999) The solutions of the Diophantine equation  $x^2 + (x + 1)^2 = y^2$  are given by

$$x = \frac{\sqrt{\frac{1}{2}(\sqrt{8B^2 + 1} - 1)} - 1}{2} \quad (2.6)$$

$$y = \frac{1}{2} \sqrt{1 + \sqrt{8B^2 + 1}} \quad (2.7)$$

when  $B$  is an odd balancing number.

Chailangka and Pakapongpun (2021) published results about neo balancing numbers from studying a Diophantine equation in analogy to balancing numbers. Consequently, many properties of neo balancing numbers and some relationships between neo balancing numbers and balancing numbers were discovered as follows.

### Neo balancing numbers

**Definition 2.20.** (Chailangka & Pakapongpun, 2021) A positive integer  $n$  is called a neo balancing number if  $n$  is a solution of the Diophantine equation

$$1 + 2 + 3 + \dots + (n - 1) = (n - 1) + (n + 0) + (n + 1) + (n + 2) + \dots + (n + r) \quad (2.8)$$

for some integer  $r \geq -1$ , which is called a neo balancer corresponding to the neo balancing number  $n$ .

For example, 2, 7, 36 and 205 are neo balancing numbers with neo balancers -1, 1, 13 and 83, respectively. We can rewrite (2.8) into

$$(n - 1)^2 = \frac{(n + r)(n + r + 1)}{2} \quad (2.9)$$

or

$$r = \frac{-(2n + 1) + \sqrt{8(n - 1)^2 + 1}}{2}. \quad (2.10)$$

From (2.9), we obtain that  $n$  is a neo balancing number if and only if  $(n - 1)^2$  is a triangular number. Also, by the given (2.10), we have  $n$  is a neo balancing number if and only if  $8(n - 1)^2 + 1$  is a perfect square. Additionally, the  $n^{\text{th}}$  neo balancing number is denoted by  $P_n$ , which satisfies a recurrence relation  $P_n = 3(P_{n-1} - 1) + 1 + \sqrt{8(P_{n-1} - 1)^2 + 1}$ . The  $n^{\text{th}}$  neo balancer is denoted by  $E_n$  and the  $n^{\text{th}}$  Lucas-neo balancing number is defined by  $Q_n := \sqrt{8(P_n - 1)^2 + 1}$ . Set initial values  $P_1 = 2$  and  $P_2 = 7$  and so on.

**Theorem 2.21.** (Chailangka & Pakapongpun, 2021) For any neo balancing number  $x$ ,  $p(x) = 3(x - 1) + 1 + \sqrt{8(x - 1)^2 + 1}$  and  $(p \circ p \circ \dots \circ p)(x)$  are also neo balancing numbers.

**Theorem 2.22.** (Chailangka & Pakapongpun, 2021) If  $x$  is a neo balancing number, then the next neo balancing number is

$$p(x) = 3(x - 1) + 1 + \sqrt{8(x - 1)^2 + 1} \quad (2.11)$$

and consequently, the previous one is

$$p^{-1}(x) = 3(x - 1) + 1 - \sqrt{8(x - 1)^2 + 1}. \quad (2.12)$$

**Theorem 2.23.** (Chailangka & Pakapongpun, 2021) Let  $B_n$  be the  $n^{\text{th}}$  balancing number,  $P_n$  the  $n^{\text{th}}$  neo balancing number,  $\overline{P_n} = P_n - 1$  and  $1 \leq k \leq n$  for any positive integers  $n$  and  $k$ . Then we have the following relations.

- (a)  $P_{n+1}P_{n-1} = (P_n + 5)(P_n - 1)$ .
- (b)  $P_{n+1}P_{n-1} + 9 = (P_n + 2)^2$ .
- (c)  $P_n = B_k \overline{P_{n-k}} - B_{k-1} \overline{P_{n-k-1}} + 1$ .
- (d)  $P_n = B_{n-1} + 1$ .

- (e)  $P_n \cdot B_n = P_n P_{n+1} - P_n$ .
- (f)  $P_n = \overline{P_{k+1}} \cdot \overline{P_{n-k}} - \overline{P_k} \cdot \overline{P_{n-k-1}} + 1$ .
- (g)  $P_{2n+1} = \overline{P_{n+1}}^2 - \overline{P_n}^2 + 1$ .
- (h)  $P_{2n} = \overline{P_n}(\overline{P_{n+1}} - \overline{P_{n-1}}) + 1$ .

**Theorem 2.24.** (Chailangka & Pakapongpun, 2021) If  $P_n$  is the  $n^{\text{th}}$  neo balancing number, then its Binet form is

$$P_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} + 1; n = 0, 1, 2, 3, \dots$$

where  $\lambda_1 = 3 + \sqrt{8}$  and  $\lambda_2 = 3 - \sqrt{8}$ .

**Theorem 2.25.** (Chailangka & Pakapongpun, 2021) If  $a$  and  $b$  are natural numbers and  $a > b$ , then

$$\overline{P_{a+b}} \cdot \overline{P_{a-b}} = (\overline{P_a} - \overline{P_b})(\overline{P_a} + \overline{P_b}).$$

**Theorem 2.26.** (Chailangka & Pakapongpun, 2021) If  $a$  and  $b$  are natural numbers and  $a \neq b$ , then

$$\frac{\overline{P_{a+b}}}{\overline{P_{a-b}}} = \frac{(\overline{P_a} - \overline{P_b})(\overline{P_a} + \overline{P_b})}{\overline{P_{a-b}}^2}$$

**Theorem 2.27.** (Chailangka & Pakapongpun, 2021) If  $a$  and  $b$  are natural numbers and  $a \neq b$ , then

$$(\overline{P_{a+b}} + \overline{P_{a-b}})^n = \sum_{r=0}^n \left[ \frac{\binom{n}{r} \left[ \sum_{s=0}^n \binom{n}{s} \overline{P_a}^n (-\overline{P_b})^{n-s} \right] \left[ \sum_{t=0}^n \binom{n}{t} \overline{P_a}^n (\overline{P_b})^{n-t} \right]}{\overline{P_{a-b}}^r} \right]$$

for some integers  $s, t \geq 0$ .

**Theorem 2.28.** (Chailangka & Pakapongpun, 2021) If  $n$  is a natural number, then

- (a)  $\overline{P_1} + \overline{P_3} + \overline{P_5} + \dots + \overline{P_{2n-1}} = \overline{P_n}^2$ .
- (b)  $\overline{P_2} + \overline{P_4} + \overline{P_6} + \dots + \overline{P_{2n}} = \overline{P_n} \cdot \overline{P_{n+1}}$ .
- (c)  $\overline{P_1} + \overline{P_2} + \overline{P_3} + \dots + \overline{P_{2n}} = \overline{P_n}(\overline{P_n} + \overline{P_{n+1}})$ .

**Theorem 2.29.** (Chailangka & Pakapongpun, 2021) If  $n$  and  $r$  are natural numbers, then

$$(Q_n + \sqrt{8P_n})^r = Q_{nr} + \sqrt{8P_{nr}}.$$

**Corollary 2.30.** (Chailangka & Pakapongpun, 2021) If  $n$  and  $r$  are natural numbers, then

$$(Q_n - \sqrt{8P_n})^r = Q_{nr} - \sqrt{8P_{nr}}.$$

**Theorem 2.31.** (Chailangka & Pakapongpun, 2021) If  $m$  and  $n$  are natural numbers, then

$$\overline{P_{m+n}} = Q_m \overline{P_n} + Q_n \overline{P_m}.$$

**Corollary 2.32.** (Chailangka & Pakapongpun, 2021) If  $m$  and  $n$  are natural numbers, then

$$Q_{m+n} = Q_m Q_n + 8 \overline{P_m} \overline{P_n}.$$

**Corollary 2.33.** (Chailangka & Pakapongpun, 2021) If  $m$  is a natural number, then

$$\overline{P_{2m}} = 2 \overline{P_m} Q_m.$$

**Lemma 2.34.** (Chailangka & Pakapongpun, 2021) If  $n$  is a natural number, then

$$(\overline{P_n}, Q_n) = 1.$$

**Lemma 2.35.** (Chailangka & Pakapongpun, 2021) If  $n$  and  $k$  are natural numbers, then

$$\overline{P_k} \text{ divides } \overline{P_{nk}}.$$

**Lemma 2.36.** (Chailangka & Pakapongpun, 2021) If  $n$  and  $k$  are natural numbers, then

$$(\overline{P_k}, Q_{nk}) = 1.$$

**Lemma 2.37.** (Chailangka & Pakapongpun, 2021) If  $n$  and  $k$  are natural numbers and  $\overline{P_k}$  divides  $\overline{P_n}$ , then  $k$  divides  $n$ .

**Theorem 2.38.** (Chailangka & Pakapongpun, 2021) If  $m$  and  $n$  are natural numbers, then

$$\overline{P}_m \text{ divides } \overline{P}_n \text{ if and only if } m \text{ divides } n.$$

**Theorem 2.39.** (Chailangka & Pakapongpun, 2021) If  $m$  and  $n$  are natural numbers, then

$$(\overline{P}_m, \overline{P}_n) = \overline{P}_{(m,n)}.$$

Furthermore, Panda introduced sequence balancing and cobalancing numbers which generalizes the concept of balancing and cobalancing numbers to an arbitrary sequence. Then, many interesting properties of these numbers were provided.

### Sequence balancing and cobalancing numbers

Throughout this section  $B_n$  is the  $n^{\text{th}}$  balancing number,  $R_n$  is the  $n^{\text{th}}$  balancer,  $b_n$  is the  $n^{\text{th}}$  cobalancing number and  $r_n$  is the  $n^{\text{th}}$  cobalancer, where  $n \in \mathbb{Z}^+$ .

**Definition 2.40.** (Panda, 2007) Let  $\{a_n\}_{n=1}^{\infty}$  be a sequence of real numbers. A number  $a_m$  of this sequence is called a sequence balancing number if

$$a_1 + a_2 + \cdots + a_{m-1} = a_{m+1} + a_{m+2} + \cdots + a_{m+r}$$

for some natural number  $r$ . Similarly, a number  $a_m$  of this sequence is called a sequence cobalancing number if

$$a_1 + a_2 + \cdots + a_m = a_{m+1} + a_{m+2} + \cdots + a_{m+r}$$

for some natural number  $r$ .

**Theorem 2.41.** (Panda, 2007) Let  $B_n$  be the  $n^{\text{th}}$  balancing number,  $b_n$  be the  $n^{\text{th}}$  cobalancing number and  $r_n$  be the  $n^{\text{th}}$  cobalancer. The sequence of sequence balancing numbers in the sequence of odd natural numbers is given by  $\{B_{n+1} + B_n\}_{n+1}^{\infty}$ , i.e.,

$$2b_{n+1} + 2r_{n+1} + 1 = B_{n+1} + B_n \text{ for } n = 1, 2, 3, \dots$$

**Theorem 2.42.** (Panda, 2007) Let  $x_n$  be the  $n^{\text{th}}$  sequence balancing number in the sequence of odd natural numbers. The sequence  $\{x_n\}_{n=1}^{\infty}$  satisfies the recurrence relation  $x_{n+1} = 6x_n - x_{n-1}$  for  $n \geq 2$ .

**Theorem 2.43.** (Panda, 2007) There does not exist any sequence cobalancing numbers in the sequence of odd natural numbers.

**Theorem 2.44.** (Panda, 2007) If  $z_n$  denotes the  $n^{\text{th}}$  sequence balancing number of the sequence  $a_n = n + 1$ , then  $z_{2n-1} = B_{n+1} - 4B_n$  and  $z_{2n} = B_{n+1} - 2B_n$  for  $n = 1, 2, 3, \dots$

**Theorem 2.45.** (Panda, 2007) There does not exist any sequence balancing number in the Fibonacci sequence.

**Theorem 2.46.** (Panda, 2007) The only sequence cobalancing number in the Fibonacci sequence is  $F_2 = 1$ .

**Theorem 2.47.** (Panda, 2007) There does not exist any balancing cube or cobalancing cube.

Lastly, Dash and Ota introduced  $t$ -balancing numbers whereby adding variable  $t$  to the Diophantine equation of balancing numbers. Then balancing numbers were generated in an innovative way and the results on  $t$ -balancing numbers were established.

### **t-Balancing numbers**

**Definition 2.48.** (Dash and Ota, 2012) Let  $t$  be a natural number. A positive integer  $n$  is called a  $t$ -balancing number if

$$1 + 2 + 3 + \dots + n = (n + 1 + t) + (n + 2 + t) + \dots + (n + r + t)$$

for some positive integer  $r$ , called the  $t$ -balancer. We denote the  $n^{\text{th}}$   $t$ -balancer corresponding to the  $n^{\text{th}}$   $t$ -balancing number  $B_n^t$  by  $R_n^t$ .

**Theorem 2.49.** (Dash and Ota, 2012) For  $n \in \mathbb{N}$ ,  $n$  is a  $t$ -balancing number if  $\sqrt{8r^2 + 8rt + 1}$  is a perfect square.

**Theorem 2.50.** (Dash and Ota, 2012) If  $n$  is natural number, then  $B_n^t = 6B_{n-2}^t - B_{n-4}^t + 2(t+1)$ ,  $t \geq 2$  is the generalized recurrence relation of  $t$ -balancing number.

**Theorem 2.51.** (Dash and Ota, 2012) If  $x$  is a  $t$ -balancing number, then  $f(x) = 3x + (t+1) + \sqrt{8x^2 + 8x(1+t) + (2t+1)^2}$  is also a  $t$ -balancing number.

**Theorem 2.52.** (Dash and Ota, 2012) If  $x$  is the  $n^{\text{th}}$   $t$ -balancing number, then the  $(n+2)^{\text{th}}$   $t$ -balancing number is

$$f(x) = 3x + (t+1) + \sqrt{8x^2 + 8x(1+t) + (2t+1)^2},$$

and consequently, the  $(n-2)^{\text{th}}$   $t$ -balancing number is

$$f(x) = 3x - (t+1) + \sqrt{8x^2 + 8x(1+t) + (2t+1)^2}.$$

**Theorem 2.53.** (Dash and Ota, 2012) For  $n, t \in \mathbb{N}$  such that  $n \geq 3$ ,

$$[B_n^t - (t+1)]^2 - B_{n+2}^t B_{n-2}^t = (2t+1)^2.$$

**Theorem 2.54.** (Dash and Ota, 2012) The Lucas  $t$ -balancing number  $c_n^t$ , satisfy the relation  $c_{n+2}^t = 6c_n^t - c_{n-2}^t$ , where  $c_n^t = A\alpha^n + B\beta^n$ , with  $\alpha = \sqrt{3+2\sqrt{2}}$  and  $\beta = \sqrt{3-2\sqrt{2}}$ .

**Theorem 2.55.** (Dash and Ota, 2012) The recurrence relation for  $t$ -balancer is,

$$R_{n+2}^t = 6R_n^t - R_{n-2}^t + 2t.$$

## Literature Reviews

Panda and Ray (2005) defined and introduced cobalancing numbers and cobalancers. The properties of cobalancing numbers were investigated and a link between the Pythagorean triplets and the cobalancing numbers was also discovered.

Panda (2006) generalized the concept of balancing and cobalancing numbers to an arbitrary sequence. Then sequence balancing numbers and sequence cobalancing

numbers are defined and demonstrated. In addition, the properties are proven that there does not exist any sequence balancing number in the Fibonacci sequence and the only sequence cobalancing number in the Fibonacci sequence is  $F_2 = 1$ . Moreover, higher order balancing and cobalancing numbers also are introduced and it is proven that there do not exist any balancing cube or cobalancing cube. Lastly, there is a conjecture on the nonexistence of balancing and cobalancing numbers of order  $k$  for  $k \geq 2$ .

Liptai, Luca, Pintér and Szalay (2009) studied balancing numbers in an innovative way. A  $(k, l)$ -balancing number was defined and investigated. Some effective and ineffective finiteness statements for the balancing numbers were proved via certain Baker-type Diophantine results and Bilu-Tichy theorem, respectively.

Panda and Ray (2011) discovered links of balancing and cobalancing numbers with Pell and associated Pell numbers. The  $n^{\text{th}}$  balancing number was proved to be the product of the  $n^{\text{th}}$  Pell number and the  $n^{\text{th}}$  associated Pell number. At the same time, the sequences of balancing and cobalancing numbers were investigated together with the relationship among Pell sequence, sequence of Lucas-balancing and Lucas-cobalancing numbers constituted associated Pell sequence.

Ray (2012) obtained the factorization of balancing and Lucas-balancing numbers using orthogonal polynomials, especially Chebyshev polynomials of first and second kind, with knowledge of number theory and linear algebra.

Rayaguru, Davala and Panda (2020) investigated balancing numbers in a new way. They studied even-odd balancing number  $n$ , which satisfy the Diophantine equation

$$2 + 4 + \cdots + (2n - 2) = (2n + 1) + (2n + 3) + \cdots + (2n + 2r - 1)$$

and call  $r$  the even-odd balancer  $r$  corresponding to the even-odd balancing number  $n$ . Then many fascinating relations with the sequence of Pell, associated Pell, balancing and cobalancing numbers also were investigated. Furthermore, some ratio formulas and identities were examined.

Chinram, Petchkaew and Hangsawat (2022) defined and introduced the con-

cept of 2-distance balancing numbers by studying Diophantine equation

$$1 + 2 + 3 + \cdots + (n - 2) = (n + 2) + (n + 3) + \cdots + (n + r).$$

The properties of recurrence relation for 2-distance balancing number were studied and the generating function and Binet formula were also discovered.

Bartz, Dearden and Iiams (2022) generalized the concept of polygonal balancing numbers and studied  $s$ -agonal balancing numbers, where companion equation is Pell-like. Then the remarkable result concerning the jump size  $j(t)$  was contributed.

Bartz, Dearden, Iiams and Metzger (2022) presented a variation called ratio balancing numbers which generalizes the sums considered and concerns an integral ratio condition. Often ratio balancing numbers possess the familiar properties of balancing numbers. However, a distinct part of ratio balancing numbers is that they exist in finite numbers for specific choices of parameters. Computational evidence directs us to conjecture that for any integer  $d$ , there are choices of parameters that yield finitely many, but at least  $d$ , ratio balancing numbers.

Hulku and Deveci (2023) introduced the Tribonacci-type balancing numbers using Diophantine equation

$$i + i^2 + i^3 + \cdots + i^{n-1} = i^{n+1} + i^{n+2} + \cdots + i^{n+k}$$

with  $i = \sqrt{-1}$ . Then the miscellaneous properties were given. The Tribonacci-type balancing sequence modulo  $m$  are also investigated and they obtain some important results involving the periods of the Tribonacci-type balancing sequences for any  $m$ . Furthermore, the multiplicative orders of the generating matrices of the Tribonacci-type balancing numbers were used to construct cyclic groups.

## CHAPTER 3

### RESEARCH METHODOLOGY

In this thesis we do the following process.

1. We study behavior equations of balancing numbers, t-balancing numbers, neo balancing numbers and others.
2. We make conjectures about t-neo balancing numbers based on known results about neo balancing numbers.
3. We prove our conjectures and make a conclusion.

## CHAPTER 4

### RESULTS

#### 4.1 Linear sequence on sequence neo balancing numbers

##### 4.1.1 Sequence neo balancing numbers

Let  $\{a_m\}_{m=1}^{\infty}$  be a sequence of real numbers. In this section, we introduce neo balancing numbers in the sequence  $\{a_m\}_{m=1}^{\infty}$ . In 2021, the neo balancing numbers and their associated properties were investigated. This dissertation re-examines the topic from the perspective of sequences.

**Definition 4.1.** Let  $\{a_m\}_{m=1}^{\infty}$  be a sequence of real numbers. A number  $a_m$  is called a sequence neo balancing number if there exists an integer  $r \geq -1$  such that

$$a_1 + a_2 + a_3 + \cdots + a_{m-1} = a_{m-1} + a_m + a_{m+1} + \cdots + a_{m+r}. \quad (4.1)$$

##### 4.1.2 Linear sequence neo balancing numbers

In this section, we will delve into the linear sequence  $\{a_m\}_{m=1}^{\infty} = \{2m + k\}_{m=1}^{\infty}$  for neo balancing numbers.

**Definition 4.2.** The linear sequence neo balancing numbers

For  $k \in \mathbb{Z}$ , consider a sequence  $\{a_m\}_{m=1}^{\infty} = \{2m + k\}_{m=1}^{\infty}$ . A number  $a_m = 2m + k$  is called a linear sequence neo balancing number if  $a_m$  is a solution to the Diophantine equation

$$a_1 + a_2 + a_3 + \cdots + a_{m-1} = a_{m-1} + a_m + a_{m+1} + \cdots + a_{m+r}, \quad (4.2)$$

for some integer  $r$ .

**Example 4.3.** If  $k = -1$ , then we have  $a_m = 2m - 1$ . We obtain the following results.

$$a_1 = 1.$$

This gives  $m = 2$ ,  $r = -1$ , and  $a_m = 3$ .

$$a_1 + a_2 + a_3 + a_4 = 1 + 3 + 5 + 7 = 16 = 7 + 9 = a_4 + a_5$$

This gives  $m = 5$ ,  $r = 0$ , and  $a_m = 9$ .

$$\begin{aligned} a_1 + a_2 + a_3 + \cdots + a_{120} &= a_{120} + a_{121} + a_{122} + \cdots + a_{169} \\ 1 + 3 + 5 + \cdots + 239 &= 239 + 241 + 243 + \cdots + 337 \\ 14400 &= 14400 \end{aligned}$$

This gives  $m = 22$ ,  $r = 7$ , and  $a_m = 43$ .

Therefore, 3, 9, and 43 are linear sequence neo balancing number.

Table 1. Examples of linear sequence neo balancing numbers

$n$	1	2	3	4	5	6
$m$	2	5	22	121	698	4061
$r$	-1	0	7	48	287	1680
$a_m = 2m - 1$	3	9	43	241	1396	8121

**Theorem 4.4.** Let  $k$  be an integer. There exists a positive integer  $m$ , such that a number  $a_m = 2m + k$  is a linear sequence neo balancing number if and only if  $8r^2 + 24r + k^2 + 2k + 17$  is a perfect square for some integer  $r$ . In fact

$$m = \frac{2r + 5 - k + \sqrt{8r^2 + 24r + k^2 + 2k + 17}}{2}. \quad (4.3)$$

*Proof.* From equation (4.2), we substitute  $a_m = 2m + k$  and obtain

$$(2 + k) + (4 + k) + \cdots + (2m - 2 + k) = (2m - 2 + k) + (2m + k) + \cdots + (2m + 2r + k). \quad (4.4)$$

$$2m = 2r + 5 - k + \sqrt{8r^2 + 24r + k^2 + 2k + 17}. \quad (4.5)$$

Since  $m$  is an index which is a positive integer, the term  $8r^2 + 24r + k^2 + 2k + 17$  needs to be perfect a square.  $\square$

Then we need to discuss some result before stating the next theorem.

Let  $y = \sqrt{8r^2 + 24r + k^2 + 2k + 17}$ . If  $x = 4r + 6$ , then

$$x^2 - 2y^2 = -2k^2 - 4k + 2 \quad (4.6)$$

$$= 4 - 2(k + 1)^2. \quad (4.7)$$

It is clear that  $(2, |k + 1|)$  is a solution of

$$x^2 - 2y^2 = 4 - 2(k + 1)^2. \quad (4.8)$$

We then use the well-known Pell's equation and the noteworthy Brahmagupta's identity.

The Pell's equation

$$x^2 - 2y^2 = 1, \quad (4.9)$$

with a fundamental solution  $(3, 2)$  is an important equation for this section. The Pell's equation (4.9) has been investigated for many years, and it is easy to find the other solutions (Rosen, 1986) by relations

$$\bar{x}_n = \frac{1}{2}[(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n] \quad (4.10)$$

$$\bar{y}_n = \frac{1}{2\sqrt{2}}[(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n]. \quad (4.11)$$

Therefore,  $(2, |k + 1|)$  is a solution to the equation (4.8) and  $(\bar{x}_n, \bar{y}_n)$  is a solution to the equation (4.9).

Table 2. Application of Brahmagupta's identity to linear sequence neo balancing numbers  $a_m = 2m + k$

	$x_j$	$y_j$	$k_j$	$x_1x_2 + 2y_1y_2$	$x_1y_2 + x_2y_1$	$k_1k_2$
$j = 1$	$\bar{x}_n$	$\bar{y}_n$	1	$2\bar{x}_n + 2 k + 1 \bar{y}_n$	$ k + 1 \bar{x}_n + 2\bar{y}_n$	$4 - 2(k + 1)^2$
$j = 2$	2	$ k + 1 $	$4 - 2(k + 1)^2$			

Then we apply the Brahmagupta's identity and obtain the triples

$$(x_1x_2 + ny_1y_2, x_1y_2 + x_2y_1, k_1k_2) \text{ and } (x_1x_2 - ny_1y_2, x_1y_2 - x_2y_1, k_1k_2).$$

Therefore, we combine equation (4.8) and equation (4.9) and their solutions. We obtain the combined equation as

$$x^2 - 2y^2 = [4 - 2(k + 1)^2](1) = 4 - 2(k + 1)^2, \quad (4.12)$$

which is analogous to equation (4.8). Additionally, we obtain the combined solution as relations

$$X_n = 2\bar{x}_n + 2|k + 1|\bar{y}_n \quad (4.13)$$

$$Y_n = |k + 1|\bar{x}_n + 2\bar{y}_n \quad (4.14)$$

and

$$X_n^* = 2\bar{x}_n - 2|k + 1|\bar{y}_n \quad (4.15)$$

$$Y_n^* = |k + 1|\bar{x}_n - 2\bar{y}_n. \quad (4.16)$$

Therefore,  $(X_n, Y_n)$  and  $(X_n^*, Y_n^*)$  are solutions to equation (4.8). We now consider recurrence relation for linear sequence neo balancing numbers.

Let  $A = 3 + 2\sqrt{2}$  and  $B = 3 - 2\sqrt{2}$ . Thus, we have

$$\begin{aligned}
 2\bar{x}_n &= A^n + B^n \\
 &= A \cdot A^{n-1} + B \cdot B^{n-1} \\
 &= (3 + 2\sqrt{2})A^{n-1} + (3 - 2\sqrt{2})B^{n-1} \\
 &= 3A^{n-1} + 2\sqrt{2}A^{n-1} + 3B^{n-1} - 2\sqrt{2}B^{n-1} \\
 &= 3(A^{n-1} + B^{n-1}) + 2\sqrt{2}(A^{n-1} - B^{n-1}).
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 2\bar{x}_{n-2} &= A^{n-2} + B^{n-2} \\
 &= \frac{A^{n-1}}{A} + \frac{B^{n-1}}{B} \\
 &= \frac{A^{n-1}}{(3 + 2\sqrt{2})} + \frac{B^{n-1}}{(3 - 2\sqrt{2})} \\
 &= (3 - 2\sqrt{2})A^{n-1} + (3 + 2\sqrt{2})B^{n-1} \\
 &= 3A^{n-1} - 2\sqrt{2}A^{n-1} + 3B^{n-1} + 2\sqrt{2}B^{n-1} \\
 &= 3(A^{n-1} + B^{n-1}) - 2\sqrt{2}(A^{n-1} - B^{n-1}).
 \end{aligned}$$

Then we obtain

$$\begin{aligned}
 2\bar{x}_n + 2\bar{x}_{n-2} &= 6(A^{n-1} + B^{n-1}) \\
 &= 6(2\bar{x}_{n-1}).
 \end{aligned}$$

Hence we find that  $\bar{x}_n$  obeys the recurrence

$$\bar{x}_n = 6\bar{x}_{n-1} - \bar{x}_{n-2}. \quad (4.17)$$

Similarly,  $\bar{y}_n$  obeys the recurrence

$$\bar{y}_n = 6\bar{y}_{n-1} - \bar{y}_{n-2}. \quad (4.18)$$

We will use the proven recurrence relation of  $\bar{x}_n$  and  $\bar{y}_n$  to find recurrence relations for  $X_n, Y_n, X_n^*$  and  $Y_n^*$ .

Since  $X_n = 2\bar{x}_n + 2|k + 1|\bar{y}_n$ , by equation (4.17) and equation (4.18), we have

$$\begin{aligned}
 X_n &= 2\bar{x}_n + 2|k + 1|\bar{y}_n \\
 &= 2(6\bar{x}_{n-1} - \bar{x}_{n-2}) + 2|k + 1|(6\bar{y}_{n-1} - \bar{y}_{n-2}) \\
 &= 6(2\bar{x}_{n-1}) - 2\bar{x}_{n-2} + 6(2|k + 1|\bar{y}_{n-1}) - 2|k + 1|\bar{y}_{n-2} \\
 &= 6(2\bar{x}_{n-1} + 2|k + 1|\bar{y}_{n-1}) - (2\bar{x}_{n-2} + 2|k + 1|\bar{y}_{n-2}) \\
 &= 6X_{n-1} - X_{n-2}.
 \end{aligned}$$

Similarly, we obtain  $Y_n = 6Y_{n-1} - Y_{n-2}$ . Although  $X_n, Y_n$  and  $X_n^*, Y_n^*$  differ slightly, we can find the recurrence relations for  $X_n^*, Y_n^*$  using similar algebraic methods as with  $X_n, Y_n$ . Therefore, we find that both pairs of sequences of  $X_n, Y_n$  and  $X_n^*, Y_n^*$  satisfy

$$X_n = 6X_{n-1} - X_{n-2} \quad (4.19)$$

$$Y_n = 6Y_{n-1} - Y_{n-2}. \quad (4.20)$$

By substituting equation (4.10), and equation (4.11) to equation (4.13), equation (4.14), equation (4.15), and equation (4.16), we get

$$\begin{aligned}
 2X_n &= (3 + 2\sqrt{2})^n(2 + |k + 1|\sqrt{2}) + (3 - 2\sqrt{2})^n(2 - |k + 1|\sqrt{2}) \\
 2\sqrt{2}Y_n &= (3 + 2\sqrt{2})^n(2 + |k + 1|\sqrt{2}) - (3 - 2\sqrt{2})^n(2 - |k + 1|\sqrt{2})
 \end{aligned}$$

and

$$\begin{aligned}
 2X_n^* &= (3 + 2\sqrt{2})^n(|k + 1|\sqrt{2} - 2) - (3 - 2\sqrt{2})^n(|k + 1|\sqrt{2} + 2) \\
 2\sqrt{2}Y_n^* &= (3 + 2\sqrt{2})^n(|k + 1|\sqrt{2} - 2) + (3 - 2\sqrt{2})^n(|k + 1|\sqrt{2} + 2).
 \end{aligned}$$

Since the sequences  $X_n, Y_n$  and  $X_n^*, Y_n^*$  sometimes make sets of solutions of the equation (4.8) for given integer  $k$ . Then we have to consider this behavior by  $X_n = |X_{n+n_0}^*|$  and  $Y_n = |Y_{n+n_0}^*|$  for some index  $n_0$ . Let  $A = 3 + 2\sqrt{2}$ ,  $B = 3 - 2\sqrt{2}$ ,  $z = 2 + |k + 1|\sqrt{2}$

and  $\bar{z} = 2 - |k + 1|\sqrt{2}$ . Thus, we obtain

$$\begin{aligned} 2X_n &= A^n z + B^n \bar{z} \\ 2X_{n+n_0}^* &= -A^{n+n_0} \bar{z} - B^{n+n_0} z. \end{aligned}$$

Since  $X_n = |X_{n+n_0}^*|$ , we have

$$(A^n z + B^n \bar{z})^2 = (2X_n)^2 = (2X_{n+n_0}^*)^2 = (A^{n+n_0} \bar{z} + B^{n+n_0} z)^2.$$

Since  $A^n z + B^n \bar{z}$  and  $A^{n+n_0} \bar{z} + B^{n+n_0} z$  are real numbers,

$$(A^n z + B^n \bar{z}) - (A^{n+n_0} \bar{z} + B^{n+n_0} z) = 0 \quad (4.21)$$

or

$$(A^n z + B^n \bar{z}) + (A^{n+n_0} \bar{z} + B^{n+n_0} z) = 0. \quad (4.22)$$

Since  $A = 3 + 2\sqrt{2}$  and  $B = 3 - 2\sqrt{2}$ , it is clear that  $AB = 1$ . Then we will consider equation (4.21);

$$\begin{aligned} (A^n z + B^n \bar{z}) - (A^{n+n_0} \bar{z} + B^{n+n_0} z) &= 0 \\ z(A^n - B^{n+n_0}) - \bar{z}(A^{n+n_0} - B^n) &= 0 \\ zB^{n_0}(A^{n+n_0} - B^n) - \bar{z}(A^{n+n_0} - B^n) &= 0 \\ (zB^{n_0} - \bar{z})(A^{n+n_0} - B^n) &= 0. \end{aligned}$$

**Lemma 4.5.** *Let  $A = 3 + 2\sqrt{2}$  and  $B = 3 - 2\sqrt{2}$ . For any non negative integers  $n, n_0$  and  $n = 0 = n_0$ ,*

$$A^{n+n_0} \neq B^{n_0}.$$

*Proof.* Let  $n, n_0$  be non negative integers with  $n = 0 = n_0$ . Suppose  $A^{n+n_0} = B^{n_0}$ .  
Then

$$n + n_0 = n_0$$

$$n = 0.$$

Substituting  $n = 0$ , we have

$$(3 + 2\sqrt{2})^{n_0} = A^{n_0} = B^{n_0} = (3 - 2\sqrt{2})^{n_0},$$

implying  $n_0 = 0$ . This is a contradiction. □

By Lemma (4.5), we have

$$zB^{n_0} - \bar{z} = 0$$

$$(2 + |k + 1|\sqrt{2})(3 - 2\sqrt{2})^{n_0} - (2 - |k + 1|\sqrt{2}) = 0,$$

but  $(3 - 2\sqrt{2})^{n_0}$  can be calculated algebraically form  $\bar{x}_{n_0} - \bar{y}_{n_0}\sqrt{2} = (3 - 2\sqrt{2})^{n_0}$  where  $x_0 = 1, y_0 = 0, x_1 = 3, y_1 = 2$ . Then

$$\begin{aligned} (2 + |k + 1|\sqrt{2})(\bar{x}_{n_0} - \bar{y}_{n_0}\sqrt{2}) - (2 - |k + 1|\sqrt{2}) &= 0 \\ 2\bar{x}_{n_0} - 2\bar{y}_{n_0}\sqrt{2} + \bar{x}_{n_0}|k + 1|\sqrt{2} - 2\bar{y}_{n_0}|k + 1| - 2 + |k + 1|\sqrt{2} &= 0 \\ \sqrt{2}(|k + 1| + \bar{x}_{n_0}|k + 1| - 2\bar{y}_{n_0}) - 2(\bar{y}_{n_0}|k + 1| - \bar{x}_{n_0} + 1) &= 0. \end{aligned}$$

Hence,

$$\bar{y}_{n_0}|k + 1| - \bar{x}_{n_0} + 1 = 0 \text{ and } |k + 1| + \bar{x}_{n_0}|k + 1| - 2\bar{y}_{n_0} = 0,$$

implying

$$|k + 1| = \frac{\bar{x}_{n_0} - 1}{\bar{y}_{n_0}} \text{ and } |k + 1| = \frac{2\bar{y}_{n_0}}{\bar{x}_{n_0} + 1}.$$

Here, we obtain 2 cases from equation (4.21) already. At this stage, we turn our attention

to equation (4.22);

$$\begin{aligned}(A^n z + B^n \bar{z}) + (A^{n+n_0} \bar{z} + B^{n+n_0} z) &= 0 \\ z(A^n + B^{n+n_0}) + \bar{z}(A^{n+n_0} + B^n) &= 0 \\ zB^{n_0}(A^{n+n_0} + B^n) + \bar{z}(A^{n+n_0} + B^n) &= 0 \\ (zB^{n_0} + \bar{z})(A^{n+n_0} + B^n) &= 0\end{aligned}$$

By Lemma (4.5), we have

$$zB^{n_0} + \bar{z} = 0.$$

By substituting algebraically form  $\bar{x}_{n_0} - \bar{y}_{n_0}\sqrt{2} = (3 - 2\sqrt{2})^{n_0}$ , we have

$$\begin{aligned}(2 + |k + 1|\sqrt{2})(3 - 2\sqrt{2})^{n_0} + (2 - |k + 1|\sqrt{2}) &= 0 \\ (2 + |k + 1|\sqrt{2})(\bar{x}_{n_0} - \bar{y}_{n_0}\sqrt{2}) + (2 - |k + 1|\sqrt{2}) &= 0 \\ 2\bar{x}_{n_0} - 2\bar{y}_{n_0}\sqrt{2} + \bar{x}_{n_0}|k + 1|\sqrt{2} - 2\bar{y}_{n_0}|k + 1| + 2 - |k + 1|\sqrt{2} &= 0 \\ \sqrt{2}(\bar{x}_{n_0}|k + 1| - |k + 1| - 2\bar{y}_{n_0}) + 2(\bar{x}_{n_0} - \bar{y}_{n_0}|k + 1| + 1) &= 0.\end{aligned}$$

Hence,

$$\bar{x}_{n_0} - \bar{y}_{n_0}|k + 1| + 1 = 0 \text{ and } \bar{x}_{n_0}|k + 1| - |k + 1| - 2\bar{y}_{n_0} = 0,$$

implying

$$|k + 1| = \frac{\bar{x}_{n_0} + 1}{\bar{y}_{n_0}} \text{ and } |k + 1| = \frac{2\bar{y}_{n_0}}{\bar{x}_{n_0} - 1}.$$

Ultimately, we obtain all 4 cases as follows.

- (i)  $|k + 1| = \frac{\bar{x}_{n_0} - 1}{\bar{y}_{n_0}}$
- (ii)  $|k + 1| = \frac{2\bar{y}_{n_0}}{\bar{x}_{n_0} + 1}$
- (iii)  $|k + 1| = \frac{\bar{x}_{n_0} + 1}{\bar{y}_{n_0}}$
- (iv)  $|k + 1| = \frac{2\bar{y}_{n_0}}{\bar{x}_{n_0} - 1}$ .

**Theorem 4.6.** Let  $A = \{-3, -2, -1, 0, 1\}$  and let  $k \in A$ . For each  $m \in \mathbb{Z}$ , let  $a_m = 2m + k$  and let  $a_{m_n}$  denote the  $n^{\text{th}}$  linear sequence neo balancing number. Then  $a_{m_n}$  satisfies

$$a_{m_n} = 6a_{m_{n-1}} - a_{m_{n-2}} - 8 \quad (4.23)$$

and the recurrence relation for the linear sequence neo balancing number's index is

$$m_n = 6m_{n-1} - m_{n-2} + 2k - 4, \quad (4.24)$$

where  $n \geq 3$ .

*Proof.* Consider the given four cases above. If we do not shift any index, then  $n_0 = 0$ , so  $\bar{x}_{n_0} = 1$ , and  $\bar{x}_{y_0} = 0$ . There is only case (ii) possible such that

$$|k + 1| = 0.$$

Hence,  $k = -1$ . Therefore,  $X_n = |X_n^*|$  and  $Y_n = |Y_n^*|$ ; in this case, both sequences share the same solution set of equation (4.8) when  $a_m = 2m - 1$ . Subsequently, if we shift an index 1 step for  $X_n = |X_{n+n_0}^*|$ , then  $n_0 = 1$ , so  $x_{n_0} = 3$ , and  $y_{n_0} = 2$ . Hence, we obtain that

$$|k + 1| = 1$$

by case (i) and case (ii). Similarly, we get

$$|k + 1| = 2$$

by case 3 and case 4. Then we have  $k = 0, -2, -3$  and  $1$ , respectively. Therefore,  $X_n = |X_{n+1}^*|$ . In other words,  $X_n$  and  $X_{n+1}^*$  have same set of solution of equation (4.8) when  $a_m = 2m, a_m = 2m - 2, a_m = 2m - 3$  and  $a_m = 2m + 1$ . That is, although the solutions to equation (4.8) used to generate the linear sequence neo balancing numbers are derived from both  $(X_n, Y_n)$  and  $(X_n^*, Y_n^*)$ , the above proof demonstrates that  $(X_n, Y_n)$  and  $(X_n^*, Y_n^*)$  share the same solution set. Consequently, the recurrence relation for the linear sequence of neo balancing numbers can be expressed as the following.

Since  $2m = 2r + 5 - k + \sqrt{8r^2 + 24r + k^2 + 2k + 17}$ ,  $y = \sqrt{8r^2 + 24r + k^2 + 2k + 17}$

and  $r = \frac{x-6}{4}$ , we get

$$2m_n = 2r + 5 - k + \sqrt{8r^2 + 24r + k^2 + 2k + 17}$$

$$2m_n = 2\left(\frac{X_n - 6}{4}\right) + 5 - k + Y_n$$

$$4m_n = X_n + 2Y_n - 2k + 4.$$

Since  $X_n$  and  $Y_n$  obey the recurrence relations (4.19) and (4.20), we have

$$\begin{aligned} 4m_n &= X_n + 2Y_n - 2k + 4 \\ &= (6X_{n-1} - X_{n-2}) + 2(6Y_{n-1} - Y_{n-2}) - 2k + 4 \\ &= 6(X_{n-1} + 2Y_{n-1} - 2k + 4) - (X_{n-2} + 2Y_{n-2} - 2k + 4) + 8k - 16 \\ &= 6(4m_{n-1}) - 4m_{n-2} + 8k - 16. \end{aligned}$$

Accordingly, we get the recurrence relation for linear sequence neo balancing number's index

$$m_n = 6m_{n-1} - m_{n-2} + 2k - 4.$$

Therefore, we have

$$\begin{aligned} a_{m_n} &= 2m_n + k \\ &= 2(6m_{n-1} - m_{n-2} + 2k - 4) + k \\ &= 6(2m_{n-1} + k) - (2m_{n-2} + k) - 8 \\ &= 6a_{m_{n-1}} - a_{m_{n-2}} - 8. \end{aligned}$$

□

**Theorem 4.7.** Let  $A = \{-3, -2, -1, 0, 1\}$ . For any integer  $k \in A^c$ , the recurrence relations for the linear sequence neo balancing numbers in  $\{a_m\} = \{2m + k\}$  are

$$a_{m_{2n-1}} = 6a_{m_{2n-3}} - a_{m_{2n-5}} - 8 \quad (4.25)$$

and

$$a_{m_{2n}} = 6a_{m_{2n-2}} - a_{m_{2n-4}} - 8, \quad (4.26)$$

where  $n \geq 3$ . Moreover, the recurrence relations for the linear sequence neo balancing number's index are

$$m_{2n-1} = 6m_{2n-3} - m_{2n-5} + 2k - 4 \quad (4.27)$$

and

$$m_{2n} = 6m_{2n-2} - m_{2n-4} + 2k - 4, \quad (4.28)$$

respectively, where  $n \geq 3$ .

*Proof.* Since  $(x_n, y_n)$  satisfies the recurrence relation

$$\bar{x}_{k+1} = 3\bar{x}_k + 4\bar{y}_k$$

$$\bar{y}_{k+1} = 2\bar{x}_k + 3\bar{y}_k,$$

we rewrite the 4 given cases.

(i)

$$\begin{aligned} |k+1| &= \frac{\bar{x}_{n_0} - 1}{\bar{y}_{n_0}} \\ &= \frac{3\bar{x}_{n_0-1} + 4\bar{y}_{n_0-1} - 1}{2\bar{x}_{n_0-1} + 3\bar{y}_{n_0-1}} \\ &= 1 + \frac{\bar{x}_{n_0-1} + \bar{y}_{n_0-1} - 1}{2\bar{x}_{n_0-1} + 3\bar{y}_{n_0-1}}. \end{aligned}$$

(ii)

$$\begin{aligned} |k+1| &= \frac{2\bar{y}_{n_0}}{\bar{x}_{n_0} + 1} \\ &= \frac{2(2\bar{x}_{n_0-1} + 3\bar{y}_{n_0-1})}{3\bar{x}_{n_0-1} + 4\bar{y}_{n_0-1} + 1} \\ &= \frac{4\bar{x}_{n_0-1} + 6\bar{y}_{n_0-1}}{3\bar{x}_{n_0-1} + 4\bar{y}_{n_0-1} + 1} \\ &= 1 + \frac{\bar{x}_{n_0-1} + 2\bar{y}_{n_0-1} - 1}{3\bar{x}_{n_0-1} + 4\bar{y}_{n_0-1} + 1}. \end{aligned}$$

(iii)

$$\begin{aligned}
|k+1| &= \frac{\bar{x}_{n_0} + 1}{\bar{y}_{n_0}} \\
&= \frac{3\bar{x}_{n_0-1} + 4\bar{y}_{n_0-1} + 1}{2\bar{x}_{n_0-1} + 3\bar{y}_{n_0-1}} \\
&= 1 + \frac{\bar{x}_{n_0-1} + \bar{y}_{n_0-1} + 1}{2\bar{x}_{n_0-1} + 3\bar{y}_{n_0-1}}.
\end{aligned}$$

(iv)

$$\begin{aligned}
|k+1| &= \frac{2\bar{y}_{n_0}}{\bar{x}_{n_0} - 1} \\
&= \frac{2(2\bar{x}_{n_0-1} + 3\bar{y}_{n_0-1})}{3\bar{x}_{n_0-1} + 4\bar{y}_{n_0-1} - 1} \\
&= \frac{4\bar{x}_{n_0-1} + 6\bar{y}_{n_0-1}}{3\bar{x}_{n_0-1} + 4\bar{y}_{n_0-1} - 1} \\
&= 1 + \frac{\bar{x}_{n_0-1} + 2\bar{y}_{n_0-1} + 1}{3\bar{x}_{n_0-1} + 4\bar{y}_{n_0-1} - 1}.
\end{aligned}$$

Ultimately, We have 4 rewritten cases:

- (i)  $|k+1| = 1 + \frac{\bar{x}_{n_0-1} + \bar{y}_{n_0-1} - 1}{2\bar{x}_{n_0-1} + 3\bar{y}_{n_0-1}}$
- (ii)  $|k+1| = 1 + \frac{\bar{x}_{n_0-1} + 2\bar{y}_{n_0-1} - 1}{3\bar{x}_{n_0-1} + 4\bar{y}_{n_0-1} + 1}$
- (iii)  $|k+1| = 1 + \frac{\bar{x}_{n_0-1} + \bar{y}_{n_0-1} + 1}{2\bar{x}_{n_0-1} + 3\bar{y}_{n_0-1}}$
- (iv)  $|k+1| = 1 + \frac{\bar{x}_{n_0-1} + 2\bar{y}_{n_0-1} + 1}{3\bar{x}_{n_0-1} + 4\bar{y}_{n_0-1} - 1}$ .

Obviously,  $|k+1|$  of the 4 above cases are not integer for  $n_0 > 1$ . Therefore,  $(X_n, Y_n)$  does not share the same solution set to equation (4.8) where  $n_0 > 1$ . Similarly, we can obtain the property as the above for  $Y_n = |Y_{n+n_0}^*|$ . Then we will obtain 2 sets of solutions to equation (4.8) which are generated by  $(X_n, Y_n)$  and  $(X_n^*, Y_n^*)$ . Therefore, we need to construct 2 recurrence relations for the line sequence neo balancing number  $\{a_m\}$  via  $(X_n, Y_n)$  and  $(X_n^*, Y_n^*)$  separately. Hence, we get the recurrence relation for linear sequence neo balancing number's index

$$m_{2n-1} = 6m_{2n-3} - m_{2n-5} + 2k - 4$$

and

$$m_{2n} = 6m_{2n-2} - m_{2n-4} + 2k - 4.$$

Since  $a_m = 2m + k$ , we get results on the generalized recurrence relation for the linear sequence neo balancing number as

$$a_{m_{2n-1}} = 6a_{m_{2n-3}} - a_{m_{2n-5}} - 8$$

and

$$a_{m_{2n}} = 6a_{m_{2n-2}} - a_{m_{2n-4}} - 8.$$

□

Examples of linear sequence neo balancing number will be provided in 4.1.3.

**Theorem 4.8.** *Let  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  be a function defined by  $f(k) = 4 - 2(k + 1)^2$ . For  $k_1, k_2 \in \mathbb{Z}$  and  $k_1 \neq k_2$ , if  $f(k_1) = f(k_2)$ , then the sequences  $\{a_m\} = \{2m + k_1\}$  and  $\{a_m\} = \{2m + k_2\}$  have the same set of linear sequence neo balancing numbers.*

*Proof.* Let  $D = 2 - (2k^2 + 4k)$  and assume  $f(k_1) = f(k_2)$ . Then we have  $D = f(k_1) = f(k_2)$ . By equation (4.8), we have

$$x^2 - 2y^2 = D = f(k_1) = f(k_2).$$

Therefore,  $k_1$  and  $k_2$  make the equation (4.8) are same. As a result, we have the same linear sequence neo balancing number by  $k_1$  and  $k_2$  if  $f(k_1) = f(k_2)$ . □

### 4.1.3 Linear sequence neo balancing numbers in some sequences

In this section, we will demonstrate linear sequence neo balancing numbers in some sequences.

**Example 4.9.** *Linear sequence neo balancing numbers in sequence  $a_m = 2m - 1$*

*By Theorem 4.6, with  $k = -1$ , the recurrence relations for the linear sequence neo*

balancing numbers  $a_m$  is

$$a_{m_n} = 6a_{m_{n-1}} - a_{m_{n-2}} - 8$$

with

$$m_n = 6m_{n-1} - m_{n-2} - 6.$$

Table 3. Linear sequence neo balancing numbers in sequence  $a_m = 2m - 1$

$n$	1	2	3	4	5	6
$m$	5	22	121	698	4061	23662
$r$	0	7	48	287	1680	9799
$a_m = 2m - 1$	9	43	241	1395	8121	47323

Note that: If we substitute  $k = -1$ , is the only solution to  $f(k) = 4$ , where  $f(k) = 4 - 2(k + 1)^2$ .

**Example 4.10.** Linear sequence neo balancing numbers in the sequences  $\{a_m\} = \{2m - 2\}$  and  $\{a_m\} = \{2m\}$

By Theorem 4.8, we can choose  $k = -2$  or  $k = 0$ . Let  $k = 0$ . By Theorem 4.6, the recurrence relations for the linear sequence neo balancing number  $a_m = 2m$  are

$$a_{m_n} = 6a_{m_{n-1}} - a_{m_{n-2}} - 8$$

and

$$m_n = 6m_{n-1} - m_{n-2} - 4.$$

Table 4. Linear sequence neo balancing numbers in sequences  $a_m = 2m - 2$  and  $a_m = 2m$

	$m$	$r$	$a_m = 2m$		$m$	$r$	$a_m = 2m - 2$
$n = 1$	1	-1	2	$n = 1$	2	-1	2
$n = 2$	2	-1	4	$n = 2$	3	-1	4
$n = 3$	7	1	14	$n = 3$	8	1	14
$n = 4$	36	13	72	$n = 4$	37	13	72
$n = 5$	205	83	410	$n = 5$	206	83	410
$n = 6$	1190	491	2380	$n = 6$	1191	491	2380

**Remark.** We can rewrite the equation of neo balancing number  $n$

$$1 + 2 + 3 + \cdots + (n - 1) = (n - 1) + (n + 0) + (n + 1) + (n + 2) + \cdots + (n + r)$$

to

$$n = \frac{2r + 5 + \sqrt{8r^2 + 24r + 17}}{2}.$$

Since  $k = 0$ , we obtain the equation (4.3) as

$$m = \frac{2r + 5 - k + \sqrt{8r^2 + 24r + k^2 + 2k + 17}}{2} = \frac{2r + 5 + \sqrt{8r^2 + 24r + 17}}{2} = n.$$

Then we have

$$a_m = 2m = 2n.$$

Similarly, if  $k = -2$ , then we obtain

$$m = \frac{2r + 7 + \sqrt{8r^2 + 24r + 17}}{2}.$$

Then we have

$$a_m = 2m - 2 = 2r + 7 + \sqrt{8r^2 + 24r + 17} - 2 = 2r + 5 + \sqrt{8r^2 + 24r + 17} = 2n.$$

Therefore, the sequences  $\{a_m\} = \{2m\}$  and  $\{a_m\} = \{2m - 2\}$  on linear sequence neo balancing numbers are twice of neo balancing numbers.

**Example 4.11.** Linear sequence neo balancing numbers in sequences  $a_m = 2m + 1$  and  $a_m = 2m - 3$

By Theorem 4.8, we can choose  $k = 1$  or  $k = -3$ . Then we choose  $k = 1$ . By Theorem 4.6, the recurrence relations for the linear sequence neo balancing number  $a_m = 2m + 1$  are

$$a_{m_n} = 6a_{m_{n-1}} - a_{m_{n-2}} - 8$$

and

$$m_n = 6m_{n-1} - m_{n-2} - 2.$$

Table 5. Linear sequence neo balancing numbers in sequence  $a_m = 2m + 1$  and  $a_m = 2m - 3$

	$m$	$r$	$a_m = 2m + 1$		$m$	$r$	$a_m = 2m - 3$
$n = 1$	2	-1	5	$n = 1$	2	-1	1
$n = 2$	9	2	19	$n = 2$	4	-1	5
$n = 3$	50	19	101	$n = 3$	11	2	19
$n = 4$	289	118	579	$n = 4$	52	19	101
$n = 5$	1682	695	3365	$n = 5$	291	118	579
$n = 6$	9801	4058	19603	$n = 6$	1684	695	3365

**Example 4.12.** Linear sequence neo balancing numbers in sequences  $a_m = 2m + 3$  and  $a_m = 2m - 5$

By Theorem 4.8, we can choose  $k = 3$  or  $k = -5$ . We choose  $k = 3$ . By Theorem 4.7, the recurrence relations for the linear sequence neo balancing number  $a_m = 2m + 3$  are

$$a_{m_{2n-1}} = 6a_{m_{2n-3}} - a_{m_{2n-5}} - 8$$

with

$$m_{2n-1} = 6m_{2n-3} - m_{2n-5} + 2$$

and

$$a_{m_{2n}} = 6a_{m_{2n-2}} - a_{m_{2n-4}} - 8$$

with

$$m_{2n} = 6m_{2n-2} - m_{2n-4} + 2$$

where  $n \geq 3$ .

Table 6. Linear sequence neo balancing numbers in sequence  $a_m = 2m + 3$  and  $a_m = 2m - 5$

	$m$	$r$	$a_m = 2m + 3$		$m$	$r$	$a_m = 2m - 5$
$n = 1$	2	-1	7	$n = 1$	2	-1	-1
$n = 2$	6	1	15	$n = 2$	2	1	-1
$n = 3$	13	4	29	$n = 3$	6	-1	7
$n = 4$	37	14	77	$n = 4$	10	1	15
$n = 5$	78	31	159	$n = 5$	17	4	29
$n = 6$	218	89	439	$n = 6$	41	14	77

## 4.2 t-Neo balancing numbers

### 4.2.1 t-Neo balancing numbers

In this section, we introduce the concept of t-neo balancing numbers and investigate some of their properties.

**Definition 4.13.** *t-Neo balancing numbers*

Let  $n$  and  $t$  be any positive integers. Then  $n$  is called a *t-neo balancing number* if  $n$  satisfies the Diophantine equation

$$1 + 2 + 3 + \dots + (n-1) = (n+t-1) + (n+t+0) + (n+t+1) + \dots + (n+t+r), \quad (4.29)$$

for some integer  $r$ . Then  $r$  is called a *t-neo balancer* which corresponding to *t-neo balancing number*  $n$ .

As an illustration of t-neo balancing number, we will demonstrate for few values of  $t$  as the following.

• 2, 7, 36, 205 and 1190 are the first five 0-neo balancing numbers with first five 0-neo balancers  $-1, 1, 13, 83$  and 491.

• 3, 15, 85, 493 and 2871 are the first five 1-neo balancing numbers with first five 1-neo balancers  $-1, 4, 33, 202, 1187$  and 16731.

• 6, 35, 204, 1189 and 6930 are the first five 2-neo balancing numbers with first five 2-neo balancers  $0, 12, 82, 490$  and 2868.

Clearly, the 0-neo balancing numbers are neo balancing numbers and 2-neo balancing numbers are balancing numbers.

**Theorem 4.14.** *Let  $n$  and  $t$  be any positive integers. If  $8n^2 + 8n(t - 2) + (2t - 3)^2$  is a perfect square, then  $n$  is a  $t$ -neo balancing number with a  $t$ -neo balancer*

$$r = \frac{-(2n + 2t + 1) \pm \sqrt{8n^2 + 8n(t - 2) + (2t - 3)^2}}{2}.$$

*Proof.* We rewrite equation (4.29) as

$$r = \frac{-(2n + 2t + 1) \pm \sqrt{8n^2 + 8n(t - 2) + (2t - 3)^2}}{2}. \quad (4.30)$$

Hence the theorem follows immediately.  $\square$

Consider equation (4.29), we have

$$n = \frac{(2r + 5) \pm \sqrt{8r^2 + 24r + 8(r + 2)t + 17}}{2}. \quad (4.31)$$

Subsequently, we will obtain the recurrence relation for  $t$ -neo balancing numbers.

**Theorem 4.15.** *Let  $n$  and  $t$  be any positive integers. If  $P_n^t$  is the  $n^{\text{th}}$   $t$ -neo balancing number, then the recurrence relation for  $P_n^t$  is*

$$P_n^t = 6P_{n-2}^t - P_{n-4}^t + 2(t - 2)$$

for  $n \geq 3$  and  $t \geq 3$ .

*Proof.* Since  $P_n^t$  is a  $t$ -neo balancing number,  $\sqrt{8r^2 + 24r + 8(r + 2)t + 17}$  is an integer. Then we can choose positive integer  $y = \sqrt{8r^2 + 24r + 8(r + 2)t + 17}$ . There-

fore, we have

$$y^2 = 8r^2 + 24r + 8(r+2)t + 17$$

$$2(2r+t+3)^2 - y^2 = 2(t-1)^2 - 1.$$

We substitute  $x = 2r + t + 3$ . Then we obtain the related Pell's equation

$$2x^2 - y^2 = 2(t-1)^2 - 1. \quad (4.32)$$

Clearly,  $x = t-1$  and  $y = 1$  is the a solution to equation (4.32). Consider Pell's equation

$$y^2 - 2x^2 = 1. \quad (4.33)$$

Note that  $(x, y) = (2, 3)$  is the fundamental solution to equation (4.33). Additionally, the algebraic forms for the other solutions  $(x_n, y_n)$  are

$$y_n + x_n\sqrt{2} = (3 + 2\sqrt{2})^n$$

$$y_n - x_n\sqrt{2} = (3 - 2\sqrt{2})^n.$$

Then we obtain

$$x_n = \frac{1}{2\sqrt{2}}((3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n)$$

$$y_n = \frac{1}{2}((3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n).$$

By Brahmagupta's identity, we know that

$$(1 - 2(t-1)^2)(y_n^2 - 2x_n^2) = (y_n \pm 2(t-1)x_n)^2 - 2(x_n \pm (t-1)y_n)^2.$$

Since  $y_n^2 - 2x_n^2 = 1$ , we have

$$2(t-1)^2 - 1 = 2((t-1)y_n \pm x_n)^2 - (2(t-1)x_n \pm y_n)^2.$$

Then we obtain 2 solutions to equation (4.32), namely

$$X_n = (t-1)y_n \pm x_n \text{ and } Y_n = 2(t-1)x_n \pm y_n.$$

Let

$$X_n^+ = (t - 1)y_n + x_n$$

$$Y_n^+ = 2(t - 1)x_n + y_n$$

and

$$X_n^- = (t - 1)y_n - x_n$$

$$Y_n^- = 2(t - 1)x_n - y_n.$$

We substitute  $x_n$  and  $y_n$  in  $X_n^+$ ,  $Y_n^+$ ,  $X_n^-$  and  $Y_n^-$  to obtain the following.

$$2\sqrt{2}X_n^+ = (3 + 2\sqrt{2})^n(\sqrt{2}(t - 1) + 1) + (3 - 2\sqrt{2})^n(\sqrt{2}(t - 1) - 1)$$

$$2Y_n^+ = (3 + 2\sqrt{2})^n(\sqrt{2}(t - 1) + 1) - (3 - 2\sqrt{2})^n(\sqrt{2}(t - 1) - 1)$$

and

$$2\sqrt{2}X_n^- = (3 + 2\sqrt{2})^n(\sqrt{2}(t - 1) - 1) + (3 - 2\sqrt{2})^n(\sqrt{2}(t - 1) + 1)$$

$$2Y_n^- = (3 + 2\sqrt{2})^n(\sqrt{2}(t - 1) - 1) - (3 - 2\sqrt{2})^n(\sqrt{2}(t - 1) + 1),$$

which individually satisfy the recurrence relation

$$X_n^\pm = 6X_{n-1}^\pm - X_{n-2}^\pm$$

$$Y_n^\pm = 6Y_{n-1}^\pm - Y_{n-2}^\pm.$$

There are two sequences of solution to equation (4.10) both  $X_n$  and  $Y_n$  such that  $X_1^+$ ,  $X_1^-$ ,  $X_2^+$ ,  $X_2^-$ ,  $X_3^+$ ,  $X_3^-$ , ... and  $Y_1^+$ ,  $Y_1^-$ ,  $Y_2^+$ ,  $Y_2^-$ ,  $Y_3^+$ ,  $Y_3^-$ , ... which are skipped by the another one and it is also true for vice versa. Then we force the recurrence relations to be  $X_n = 6X_{n-2} - X_{n-4}$  and  $Y_n = 6Y_{n-2} - Y_{n-4}$ . Since we already have the required components  $n$  from equation (4.31),  $x = 2r + t + 3$  and  $y = \sqrt{8r^2 + 24r + 8(r + 2)t + 17}$ ,

we would discover the  $n^{\text{th}}$  t-neo balancing number by

$$\begin{aligned} 2P_n^t &= (2r + 5) \pm \sqrt{8r^2 + 24r + 8(r + 2)t + 17} \\ &= 2r + t + 3 + X_n^\pm + 2 - t \\ &= X_n^\pm + Y_n^\pm - (t - 2). \end{aligned}$$

Therefore,

$$P_n^t = \frac{1}{2}(X_n^\pm + Y_n^\pm - (t - 2)).$$

Moreover, we substitute  $X_n$  and  $Y_n$  and obtain the recurrence relation for t-neo balancing numbers via recurrence relations of  $X_n$  and  $Y_n$ .

$$\begin{aligned} 2P_n^t &= X_n + Y_n - (t - 2) \\ &= 6X_{n-2} - X_{n-4} + 6Y_{n-2} - Y_{n-4} - (t - 2) \\ &= 6[X_{n-2} + Y_{n-2} - (t - 2)] - [X_{n-4} + Y_{n-4} - (t - 2)] + 6(t - 2) - (t - 2) - (t - 2) \\ &= 6(2P_{n-2}^t) - 2P_{n-4}^t + 4(t - 2). \end{aligned}$$

Therefore,

$$P_n^t = 6P_{n-2}^t - P_{n-4}^t + 2(t - 2), \text{ where } t \geq 5. \quad (4.34)$$

□

We compute the first three t-neo balancing numbers by putting value of  $n$  for proof the following theorem. Furthermore, the sequence  $\{X_n^+\}$  and  $\{Y_n^+\}$  generate odd t-neo balancing numbers, meanwhile  $\{X_n^-\}$  and  $\{Y_n^-\}$  generate even t-neo balancing numbers as shown in table 7 below.

Table 7. First three t-neo balancing numbers

$n$	$X_n^+$	$Y_n^+$	$X_n^-$	$Y_n^-$	$P_n^t(X_n^+, Y_n^+)$	$P_n^t(X_n^-, Y_n^-)$
1	$3t - 1$	$4t - 1$	$3t - 5$	$4t - 7$	$3t$	$3t - 5$
2	$17t - 5$	$24t - 7$	$17t - 29$	$24t - 41$	$20t - 5$	$20t - 34$
3	$99t - 29$	$140t - 41$	$99t - 169$	$140t - 239$	$119t - 34$	$119t - 203$

#### 4.2.2 Function Generating t-Neo Balancing Numbers

In this section, we will introduce the function which can generate the next t-neo balancing number as the following theorem.

**Theorem 4.16.** *Let  $t$  be a non negative integer and  $t \geq 3$ . If  $x$  is a t-neo balancing number, then*

$$f(x) = 3x + t - 2 + \sqrt{8x^2 + 8x(t - 2) + (2t - 3)^2} \quad (4.35)$$

*is a t-neo balancing number.*

*Proof.* Let  $u = f(x)$ . Then we have  $x < u$  and obtain that

$$x = 3u + t - 2 - \sqrt{8u^2 + 8u(t - 2) + (2t - 3)^2}.$$

Since we have known that  $x$  is a t-neo balancing number, then we would obtain that  $8u^2 + 8u(t - 2) + (2t - 3)^2$  is a perfect square. From Theorem 4.14,  $u = f(x)$  is a t-neo balancing numbers.  $\square$

**Theorem 4.17.** *If  $x$  is the  $n^{\text{th}}$  t-neo balancing number, then the  $(n+2)^{\text{th}}$  t-neo balancing number is*

$$f(x) = 3x + t - 2 + \sqrt{8x^2 + 8x(t - 2) + (2t - 3)^2} \quad (4.36)$$

*and the  $(n - 2)^{\text{th}}$  t-neo balancing number is*

$$f(x) = 3x + t - 2 - \sqrt{8x^2 + 8x(t - 2) + (2t - 3)^2} \quad (4.37)$$

*for  $n \geq 3$  and  $t \geq 3$ .*

*Proof.* Let  $t$  be a non negative integer where  $t \geq 3$ . Define the function  $f : [0, \infty) \rightarrow [3t - 5, \infty)$  by

$$f(x) = 3x + t - 2 + \sqrt{8x^2 + 8x(t - 2) + (2t - 3)^2}.$$

Since

$$f'(x) = 3 + \frac{16x + 4x(t - 2)}{2\sqrt{8x^2 + 8x(t - 2) + (2t - 3)^2}} > 0,$$

we obtain that  $f$  is a strictly increasing function.. Then  $f$  is invertible and  $x < f(x)$  for all  $x \geq 0$ . Hence  $f^{-1}$  exists  $f^{-1}(x) < x$ . Moreover, we have  $f^{-1}(x) = 3x + t - 2 -$

$\sqrt{8x^2 + 8x(t-2) + (2t-3)^2}$ . Let  $g(x) = f^{-1}(x)$ . Since

$$\begin{aligned} 8g^2(x) + 8g(x)t - 16g(x) + 4t^2 - 12t + 9 \\ = \left( 3\sqrt{8x^2 + 8x(t-2) + (2t-3)^2} - 8g(x) - 4(t-2) \right)^2, \end{aligned}$$

we obtain that  $8g^2(x) + 8g(x)t - 16g(x) + 4t^2 - 12t + 9$  is a perfect square. Thus,  $g(x) = f^{-1}(x)$  also is a t-neo balancing number. Currently, we will prove the rest of this proof by mathematical induction. Since the first three t-neo balancing numbers obtained from  $X_n^+$ , and  $Y_n^+$  are  $P_1^t = 3t$ ,  $P_2^t = 20t - 5$  and  $P_3^t = 119t - 34$  and the first three t-neo balancing numbers obtained from  $X_n^-$ , and  $Y_n^-$  are  $P_1^t = 3t - 5$ ,  $P_2^t = 20t - 34$  and  $P_3^t = 119t - 203$  which satisfy the condition  $f(P_i^t) = P_{i+1}^t$  and  $X_n^+$ , and  $Y_n^+$  generate the odd terms for t-neo balancing numbers and  $X_n^-$ , and  $Y_n^-$  generate the even terms for t-neo balancing numbers respectively. Therefore,

$$P_1^t(X_1^+, Y_1^+) = P_1^t,$$

$$P_1^t(X_1^-, Y_1^-) = P_2^t,$$

$$P_2^t(X_2^+, Y_2^+) = P_3^t,$$

$$P_2^t(X_2^-, Y_2^-) = P_4^t,$$

$$P_3^t(X_3^+, Y_3^+) = P_5^t,$$

$$P_3^t(X_3^-, Y_3^-) = P_6^t.$$

We will assume the hypothesis  $P(n)$  that there is no even (or odd) t-neo balancing number between  $P_{n-1}^t$  and  $P_n^t$ . Assume that  $P(k)$  is true. Therefore, there is no even (or odd) t-neo balancing number between  $P_{k-1}^t$  and  $P_k^t$ . Then we suppose that there exists a t-neo balancing number  $y$  between  $P_k^t$  and  $P_{k+1}^t$ . Therefore,

$$\begin{aligned} P_k^t &< y < P_{k+1}^t \\ f^{-1}(P_k^t) &< f^{-1}(y) < f^{-1}(P_{k+1}^t) \\ P_{k-1}^t &< f^{-1}(y) < P_k^t. \end{aligned}$$

Since  $f^{-1}(y)$  also is a t-neo balancing number, this is a contradiction. Then, there is no even (or odd) t-neo balancing number between  $P_{n-1}^t$  and  $P_n^t$ .  $\square$

### 4.2.3 Property of t-Neo Balancing Numbers

**Theorem 4.18.** *If  $P_n^t$  is the  $n^{\text{th}}$  t-neo balancing number, then*

$$[P_n^t - (t - 2)]^2 - P_{n+2}^t P_{n-2}^t = (2t - 3)^2.$$

for  $n \geq 3$  and  $t \geq 3$ .

*Proof.* Since  $P_{n+2}^t = 6P_n^t - P_{n-2}^t + 2(t - 2)$ , we have

$$\frac{1}{P_n^t} (P_{n+2}^t + P_{n-2}^t - 2(t - 2)) = 6.$$

Similarly, we have  $\frac{1}{P_{n-2}^t} (P_n^t + P_{n-4}^t - 2(t - 2)) = 6$ . Then

$$\begin{aligned} [P_{n+2}^t + P_{n-2}^t - 2(t - 2)](P_{n-2}^t) &= [P_n^t + P_{n-4}^t - 2(t - 2)](P_n^t) \\ P_{n+2}^t P_{n-2}^t + (P_{n-2}^t)^2 - 2P_{n-2}^t(t - 2) &= (P_n^t)^2 + P_n^t P_{n-4}^t - 2P_n^t(t - 2) \\ P_{n+2}^t P_{n-2}^t + (P_{n-2}^t)^2 - 2P_{n-2}^t(t - 2) + (t - 2)^2 &= (P_n^t)^2 + P_n^t P_{n-4}^t - 2P_n^t(t - 2) + (t - 2)^2 \\ P_{n+2}^t P_{n-2}^t + [P_{n-2}^t - (t - 2)]^2 &= [P_n^t - (t - 2)]^2 + P_n^t P_{n-4}^t. \end{aligned}$$

Therefore, we get

$$[P_{n-2}^t - (t - 2)]^2 - P_n^t P_{n-4}^t = [P_n^t - (t - 2)]^2 - P_{n+2}^t P_{n-2}^t. \quad (4.38)$$

In the same way, we obtain

$$[P_{n-4}^t - (t - 2)]^2 - P_{n-2}^t P_{n-6}^t = [P_{n-2}^t - (t - 2)]^2 - P_n^t P_{n-4}^t. \quad (4.39)$$

From equation (4.38) and (4.39), we have

$$[P_n^t - (t - 2)]^2 - P_{n+2}^t P_{n-2}^t = [P_{n-4}^t - (t - 2)]^2 - P_{n-2}^t P_{n-6}^t$$

Continuing along these lines, we obtain

$$[P_n^t - (t - 2)]^2 - P_{n+2}^t P_{n-2}^t = \begin{cases} [P_3^t - (t - 2)]^2 - P_1^t P_5^t & \text{if } n \text{ is odd} \\ [P_4^t - (t - 2)]^2 - P_2^t P_6^t & \text{if } n \text{ is even,} \end{cases} \quad (4.40)$$

where  $P_1^t = 3t$ ,  $P_3^t = 20t - 5$ ,  $P_5^t = 119t - 34$  and  $P_2^t = 3t - 5$ ,  $P_4^t = 20t - 34$ ,  $P_6^t = 119t - 203$ . If  $n$  is odd, then we have

$$\begin{aligned} [P_3^t - (t - 2)]^2 - P_1^t P_5^t &= [20t - 5 - (t - 2)]^2 - (3t)(119t - 34) \\ &= (19t - 3)^2 - 357t^2 + 102t \\ &= 361t^2 - 114t + 9 - 357t^2 + 102t \\ &= 4t^2 - 12t + 9 \\ &= (2t - 3)^2. \end{aligned}$$

If  $n$  is even, then we have

$$\begin{aligned} [P_4^t - (t - 2)]^2 - P_2^t P_6^t &= [20t - 34 - (t - 2)]^2 - (3t - 5)(119t - 203) \\ &= (19t - 32)^2 - (357t^2 - 1204t + 105) \\ &= 361t^2 - 1216t + 1024 - (357t^2 - 1204t + 105) \\ &= 4t^2 - 12t + 9 \\ &= (2t - 3)^2. \end{aligned}$$

Therefore,

$$[P_n^t - (t - 2)]^2 - P_{n+2}^t P_{n-2}^t = (2t - 3)^2. \quad (4.41)$$

□

**Definition 4.19.** *Lucas  $t$ -neo balancing numbers*

For  $t, n \geq 3$ , if  $P_{n+2}^t = 3P_n^t + t - 2 + \sqrt{8(P_n^t)^2 + 8P_n^t(t - 2) + (2t + 3)^2}$  is the  $(n + 2)^{\text{th}}$   $t$ -neo balancing number, then the  $n^{\text{th}}$  Lucas  $t$ -neo balancing number is

$$c_n^t = \sqrt{8(P_n^t)^2 + 8P_n^t(t - 2) + (2t + 3)^2}. \quad (4.42)$$

**Theorem 4.20.** *The Lucas  $t$ -neo balancing numbers satisfy  $c_{n+2}^t = 6c_n^t - c_{n-2}^t$  where  $n \geq 3$  and  $t \geq 3$ .*

*Proof.* Since  $c_n^t = \sqrt{8(P_n^t)^2 + 8P_n^t(t-2) + (2t+3)^2}$ , we have

$$c_{n+2}^t = \sqrt{8(P_{n+2}^t)^2 + 8P_{n+2}^t(t-2) + (2t-3)^2}.$$

Since  $P_{n+2}^t = 3P_n^t + t - 2 + \sqrt{8(P_n^t)^2 + 8P_n^t(t-2) + (2t-3)^2}$ , we have

$$P_{n+2}^t = 3P_n^t + t - 2 + c_n^t. \quad (4.43)$$

Then

$$\begin{aligned} 8(P_{n+2}^t)^2 &= 8(3P_n^t + t - 2 + c_n^t)^2 \\ &= 72(P_n^t)^2 + 8(t-2)^2 + 8(c_n^t)^2 + 48P_n^t(t-2) + 48P_n^t c_n^t + 16c_n^t(t-2). \end{aligned}$$

By Theorem 4.18, we have

$$\begin{aligned} (2t-3)^2 &= [P_n^t - (t-2)]^2 - P_{n+2}^t P_{n-2}^t \\ &= [P_n^t - (t-2)]^2 - (3P_{n+2}^t + t - 2 + R_{n+2}^t)(3P_{n-2}^t + t - 2 + R_{n-2}^t) \\ &= -8(P_n^t)^2 + (c_n^t)^2 - 8P_n^t(t-2). \end{aligned}$$

Then we combine them and obtain that

$$c_{n+2}^t = 3c_n^t + 8P_n^t + 4(t-2). \quad (4.44)$$

Similarly, we will consider  $c_{n-2}^t$ . Then we will get

$$c_{n-2}^t = 3c_n^t - 8P_n^t - 4(t-2). \quad (4.45)$$

Combining equation (4.44) and (4.45) yields

$$c_{n+2}^t = 6c_n^t - c_{n-2}^t.$$

□

**Theorem 4.21.** If  $c_n^t$  is the  $n^{\text{th}}$  Lucas  $t$ -neo balancing number,  $c_n^t = A\alpha^n + B\beta^n$ , with  $\alpha = \sqrt{3 + 2\sqrt{2}}$ ,  $\beta = \sqrt{3 - 2\sqrt{2}}$  for  $n \geq 3$  and  $t \geq 3$ .

*Proof.* Since the recurrence relation for  $c_n^t$  is a fourth order homogeneous recurrence relation, we have the characteristic equation

$$m^4 - 6m^2 + 1 = 0.$$

Let  $y = m^2$ . We get

$$y^2 - 6y + 1 = 0.$$

Therefore,

$$\begin{aligned} y &= \frac{6 \pm \sqrt{32}}{2} \\ &= 3 \pm 2\sqrt{2}. \end{aligned}$$

Thus,  $m = \sqrt{3 \pm 2\sqrt{2}}$ . Hence, we obtain the solution  $c_n^t = A\alpha^n + B\beta^n$  for a homogeneous recurrence relation, where  $\alpha = \sqrt{3 + 2\sqrt{2}}$ ,  $\beta = \sqrt{3 - 2\sqrt{2}}$ .  $\square$

**Theorem 4.22.** The recurrence relation of  $t$ -neo balancer is

$$R_{n+2}^t = 6R_n^t - R_{n-2}^t + 2t$$

for  $n \geq 3$  for  $t \geq 3$ .

*Proof.* By equation (4.30), we have

$$R_n^t = r = \frac{-2P_n^t - 2t - 1 + c_n}{2}, \quad (4.46)$$

where  $R_n^t$  denote the  $n^{\text{th}}$   $t$ -neo balancer. Then, by equation (4.46), equation (4.45), and

equation (4.43) we have

$$\begin{aligned} 2R_{n-2}^t &= -2P_{n-2}^t - 2t - 1 + c_{n-2}^t, \\ c_{n-2}^t &= 3c_n^t - (8P_n^t + 4(t-2)), \\ P_{n-2}^t &= 3P_n^t + (t-2) - c_n^t, \end{aligned}$$

and

$$\begin{aligned} 2R_{n+2}^t &= -2P_{n+2}^t - 2t - 1 + c_{n+2}^t, \\ c_{n+2}^t &= 3c_n^t + (8P_n^t + 4(t-2)), \\ P_{n+2}^t &= 3P_n^t + (t-2) + c_n^t. \end{aligned}$$

Then,

$$2R_{n-2}^t = -14P_n^t - 6(t-2) + 5c_n^t - 2t - 1 \quad (4.47)$$

$$2R_{n+2}^t = 2P_n^t + 2(t-2) + c_n^t - 2t - 1. \quad (4.48)$$

From equation (4.47) and equation (4.48), we have

$$\begin{aligned} 2R_{n-2}^t + 2R_{n+2}^t &= -12P_n^t - 4(t-2) + 6c_n^t - 4t - 2 \\ &= 2R_n^t + 4t + 12. \end{aligned}$$

Therefore,

$$R_{n+2}^t = 6R_n^t - R_{n-2}^t + 2t + 6.$$

□

### 4.3 Sequence t-Neo balancing numbers

#### 4.3.1 Sequences on Sequence t-Neo Balancing numbers from certain sequences

In this section, we introduce combinations of sequence and t-neo balancing numbers which are called sequence t-neo balancing numbers.

**Definition 4.23.** *Sequence t-neo balancing numbers*

Let  $\{a_m\}_{m=1}^{\infty}$  be a real sequence and  $t \in \mathbb{Z}^+$ . The number  $a_m$  is called a sequence t-neo balancing number if

$$a_1 + a_2 + a_3 + \cdots + a_{m-1} = a_{m+t-1} + a_{m+t} + a_{m+t+1} + \cdots + a_{m+t+r}, \quad (4.49)$$

for some integer  $r \geq -1$ .

**4.3.2 Sequence t-neo balancing numbers associate to the sequence  $\{a_m\} = \{2m - 1\}$**

**Theorem 4.24.** *Let t be any positive integers. Then there exists a positive integer m, such that a number  $a_m = 2m - 1$  is a sequence t-neo balancing number if and only if  $8r^2 + 24r + 8rt + 16t + 16$  is a perfect square for some integer r.*

*Proof.* Since equation (4.49), we survey a sequence t-neo balancing numbers by a certain sequence. Hence, we find a sequence  $a_m = 2m - 1$ , which is called the sequence t-neo balancing numbers if

$$1 + 3 + \cdots + (2m - 3) = (2m + 2t - 3) + (2m + 2t - 1) + \cdots + (2m + 2t + 2r - 1) \quad (4.50)$$

$$2m = 2r + 6 + \sqrt{8r^2 + 24r + 8rt + 16t + 16}. \quad (4.51)$$

Then  $8r^2 + 24r + 8rt + 16t + 16$  must be a perfect square. □

**4.3.3 Recurrence relations for sequence t-neo balancing numbers**

**Theorem 4.25.** *If  $a_m = 2m - 1$ , then the recurrence relation for the sequence t-neo balancing number's index is*

$$m_{2n-1} = 6m_{2n-3} - m_{2n-5} - 6$$

for  $n \geq 3$ . Subsequently, the recurrence relation for the sequence t-neo balancing number is

$$a_{m_{2n-1}} = 6a_{m_{2n-3}} - a_{m_{2n-5}} + 4t - 8$$

for  $n \geq 3$  and  $t \geq 3$ .

*Proof.* Since  $a_m$  is a sequence t-neo balancing number, the term

$$8r^2 + 24r + 8rt + 16t + 16$$

in equation (4.52) is a perfect square. Then we let

$$y = \sqrt{8r^2 + 24r + 8rt + 16t + 16}.$$

If we choose  $x = 4r + 6 + 2t$ , then we will obtain

$$x^2 - 2y^2 = (2(t-1))^2 - 8. \quad (4.52)$$

Therefore, we obtain the triple

$$(a, b, k) = (2(t-1), 2, (2(t-1))^2 - 8) \quad (4.53)$$

as a solution to the equation (4.52). As before, it is known that the Pell's equation

$$x^2 - 2y^2 = 1 \quad (4.54)$$

has solutions

$$\begin{aligned} \bar{x}_n &= \frac{1}{2}[(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n], \\ \bar{y}_n &= \frac{1}{2\sqrt{2}}[(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n]. \end{aligned}$$

Therefore, we have the triple  $(\bar{x}_n, \bar{y}_n, 1)$  for equation (4.54). Thus, we apply the Brahmagupta's identity to  $(2(t-1), 2, (2(t-1))^2 - 8)$  and  $(\bar{x}_n, \bar{y}_n, 1)$ .

Table 8. Application of Brahmagupta's identity to sequence t-neo balancing number

$$a_m = 2m - 1$$

$j$	$x_j$	$y_j$	$k_j$	$x_1x_2 + 2y_1y_2$	$x_1y_2 + x_2y_1$
1	$2(t-1)$	2	$(2(t-1))^2 - 8$	$2(t-1)\bar{x}_n + 2(2\bar{y}_n)$	$2(t-1)\bar{y}_n + 2\bar{x}_n$
2	$\bar{x}_n$	$\bar{y}_n$	1		

Thus, we find solutions

$$X_n^+ = 2(t-1)\bar{x}_n + 2(2\bar{y}_n)$$

$$Y_n^+ = 2(t-1)\bar{y}_n + 2\bar{x}_n$$

and

$$X_n^- = 2(t-1)\bar{x}_n - 2(2\bar{y}_n)$$

$$Y_n^- = 2(t-1)\bar{y}_n - 2\bar{x}_n,$$

of (4.52). Subsequently, we substitute  $\bar{x}_n$  and  $\bar{y}_n$ . Then we obtain

$$2X_n^+ = (3 + 2\sqrt{2})^n(2(t-1) + 2\sqrt{2}) + (3 - 2\sqrt{2})^n(2(t-1) - 2\sqrt{2})$$

$$2\sqrt{2}Y_n^+ = (3 + 2\sqrt{2})^n(2(t-1) + 2\sqrt{2}) - (3 - 2\sqrt{2})^n(2(t-1) - 2\sqrt{2})$$

and

$$2X_n^- = (3 + 2\sqrt{2})^n(2(t-1) - 2\sqrt{2}) + (3 - 2\sqrt{2})^n(2(t-1) + 2\sqrt{2})$$

$$2\sqrt{2}Y_n^- = (3 + 2\sqrt{2})^n(2(t-1) - 2\sqrt{2}) - (3 - 2\sqrt{2})^n(2(t-1) + 2\sqrt{2}).$$

Then it can be shown that  $X_n^\pm$ , and  $Y_n^\pm$  satisfy the recurrence relation

$$X_n^\pm = 6X_{n-1}^\pm - X_{n-2}^\pm$$

$$Y_n^\pm = 6Y_{n-1}^\pm - Y_{n-2}^\pm.$$

Since

$$2m = 2r + 6 + \sqrt{8r^2 + 24r + 8rt + 16t + 16},$$

$$x = 4r + 6 + 2t,$$

$$y = \sqrt{8r^2 + 24r + 8rt + 16t + 16},$$

we obtain

$$2r = \frac{x - 6 - 2t}{2}$$

and

$$\begin{aligned} 2m_n &= 2r + 6 + \sqrt{8r^2 + 24r + 8rt + 16t + 16} \\ &= \frac{X_n - 6 - 2t}{2} + 6 + Y_n. \end{aligned}$$

Then by recurrence relations for  $X_n$  and  $Y_n$ , we have

$$\begin{aligned} 4m_n &= X_n + 2Y_n + 6 - 2t \\ &= 6X_{n-1} - X_{n-2} + 2(6Y_{n-1} - Y_{n-2}) + 6 - 2t \\ &= 6(X_{n-1} + 2Y_{n-1} + 6 - 2t) - (X_{n-2} + 2Y_{n-2} + 6 - 2t) + 8t - 24 \\ &= 6(4m_{n-1}) - 4m_{n-2} + 8t - 24. \end{aligned}$$

Then we get the index's relation

$$m_n = 6m_{n-1} - m_{n-2} + 2t - 6.$$

Since  $a_m = 2m - 1$ , we obtain the recurrence relation

$$\begin{aligned} a_{m_n} &= 2m_n - 1 \\ &= 2(6m_{n-1} - m_{n-2} + 2t - 6) - 1 \\ &= 6(2m_{n-1} - 1) - (2m_{n-2} - 1) + 4t - 8 \\ &= 6a_{m_{n-1}} - a_{m_{n-2}} + 4(t - 2). \end{aligned}$$

Table 9. Examples of sequence t-neo balancing numbers in  $2m - 1$

$n$	$m_n^-(X_n^-, Y_n^-)$	$m_n^+(X_n^+, Y_n^+)$	$a_{m_n^-}$	$a_{m_n^+}$
1	$3t - 7$	$3t + 3$	$6t - 15$	$6t + 5$
2	$20t - 48$	$20t + 10$	$40t - 97$	$40t + 19$
3	$119t - 287$	$119t + 51$	$238t - 575$	$238t + 101$

Similar to section t-neo balancing numbers, there are two sequences both  $X_n, X_n^*, Y_n$  and  $Y_n^*$  such that  $X_1, X_1^*, X_2, X_2^*, X_3, X_3^*, \dots$  and  $Y_1, Y_1^*, Y_2, Y_2^*, Y_3, Y_3^*, \dots$ . We find that the sequence is skipped by another one for each. But there is  $t = 3$  such that,  $m_n$  and  $a_{m_n}$  is shifted 1 step of  $m_n^*$  and  $a_{m_n^*}$ . Nevertheless,  $t = 3$  also obey skipped recurrence.

Hence, we force the recurrence to be

$$a_{m_n} = 6a_{m_{n-2}} - a_{m_{n-4}} + 4(t - 2)$$

and the index's recurrence

$$m_n = 6m_{n-2} - m_{n-4} + 2t - 6.$$

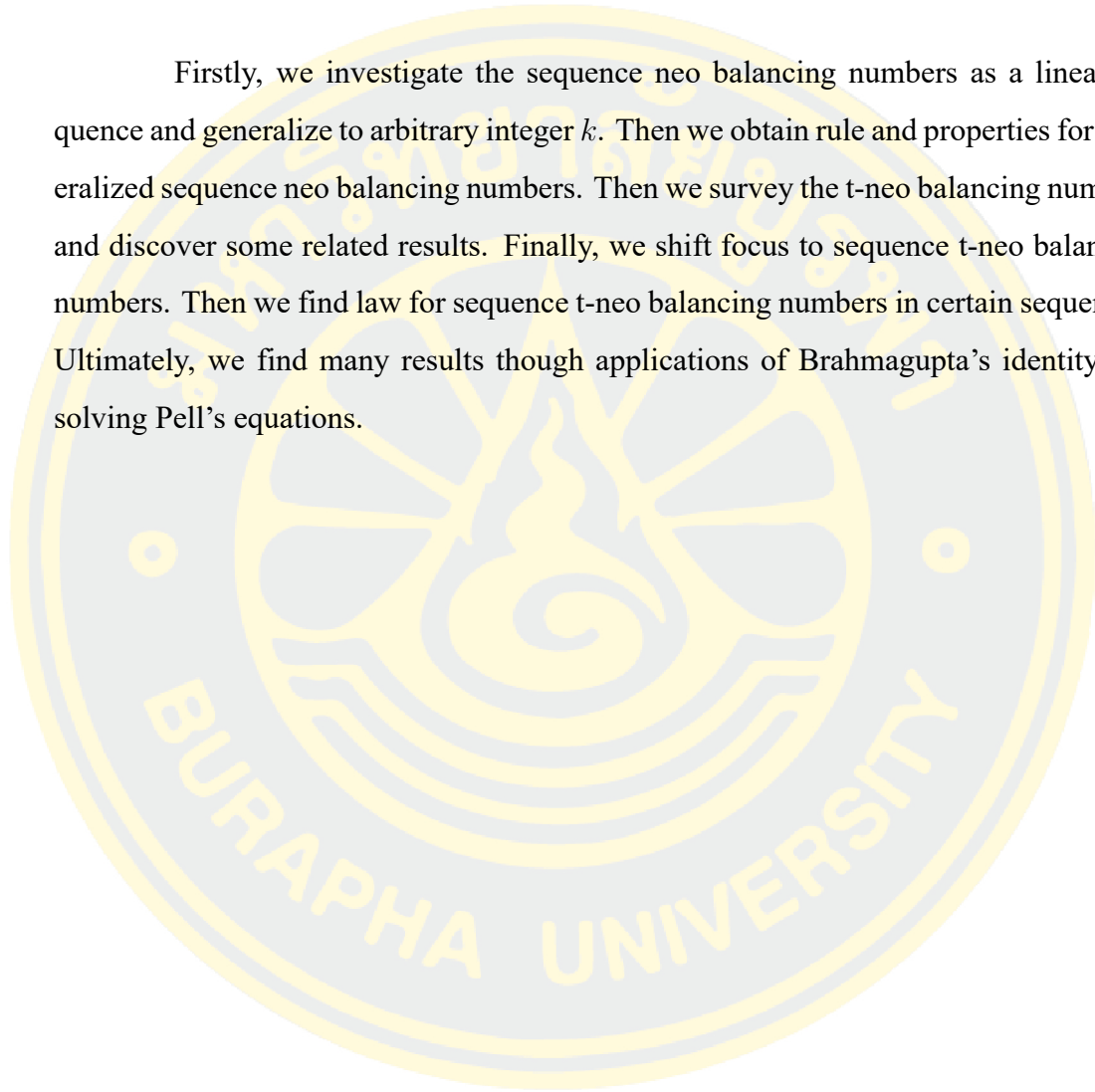
□



## CHAPTER 5

### CONCLUSION AND DISCUSSION

Firstly, we investigate the sequence neo balancing numbers as a linear sequence and generalize to arbitrary integer  $k$ . Then we obtain rule and properties for generalized sequence neo balancing numbers. Then we survey the t-neo balancing numbers and discover some related results. Finally, we shift focus to sequence t-neo balancing numbers. Then we find law for sequence t-neo balancing numbers in certain sequences. Ultimately, we find many results though applications of Brahmagupta's identity and solving Pell's equations.



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