EXISTENCE AND STABILITY ANALYSIS FOR NONLOCAL BOUNDARY VALUE PROBLEMS VIA PROPORTIONAL CAPUTO FRACTIONAL DERIVATIVE

BOUNMY KHAMINSOU

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DOCTOR DEGREE OF PHILOSOPHY
IN MATHEMATICS
FACULTY OF SCIENCE
BURAPHA UNIVERSITY
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The Dissertation of Bounmy Khaminsou has been approved by the examining committee to be partial fulfillment of the requirements for the Doctor Degree of Philosophy in Mathematics of Burapha University

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In this dissertation, we considered the existence results and stability results for a class of nonlinear fractional pantograph differential equation with mixed nonlocal boundary conditions and integro-differential Langevin equation with nonlocal fractional integral conditions via a proportional Caputo fractional derivative. We establish some sufficient conditions for the existence and uniqueness results by using some well known fixed point theorems. Finally, we give some examples support of our main results.

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Bounmy Khaminsou

CONTENTS

CHAPTER 1

INTRODUCTION

Statement of the problem

Fractional calculus ($\mathbb{FC}$) is a section of mathematical which examines properties of non-integer or complex number orders of derivatives and integrals (is said to be fractional derivatives and fractional integrals operators). In 1876, Riemann presented the definition of the Riemann-Liouville(RL) derivative. Caputo is the first one who proposed another definition of $\mathbb{FC}$ via a modified RL fractional integral at the beginning of the 20^{th} century, namely Caputo fractional derivative. In 1982 Hadamard introduced definitions and properties of Hadamard fractional derivative and integral. Caputo and Fabrizio presented a non-local derivative without a singular kernel and obtained the new Caputo-Fabrizio fractional derivative of order $0 < \alpha < 1$. In 2011, Katugampola introduced a fractional integral and fractional derivative, which generalized the RL and the Hadamard fractional operators into a single form. In 2014 Khalil et al presented another type of local fractional derivative called conformable fractional derivative. Recently, Anderson and Ulness (2015) utilized the concept of the proportional derivative controller to modify the conformable derivative. Nowadays researcher presented many type of fractional operators in fractional calculus likes RL, Caputo, Hadamard, Katugampola, proportional and many others.

Proportional fractional operator is one type of fractional operators. It can cover the Caputo operator type. Recently a lot of researchers have used the proportional fractional operator for their papers. Alzabut (2019) presented Gronwall inequalities involving generalized proportional derivative. Jarad et al. (2020) presented a general type of fractional proportional integral and derivative. Rahman et al. (2020) studied bounds of generalized proportional fractional integral in general form via convex function.

Fractional differential equations (FDE) are equations which involve a relation between one or more variables function and their fractional derivative. The fractional differential equations are the main area of research, they have used in many fields of science and engineering. Nowadays, fractional differential equations have been investigated by many reseachers.

Ulam stability was presented by Ulam in 1940. He studied the stability problem of the solutions of functional equations. In the next year, Hyers (1941) has solved stability problem, which is called Ulam-Hyers ($\mathcal{UH}$) stability. It is an important tool which was used by many researchers. Obloza Seem is the first person who has investigated $\mathcal{UH}$ stability of linear differential equations. Vanterler et al. (2017) studied $\mathcal{UH}$ stability and the Ulam-Hyers-Rassias ($\mathcal{UHR}$)stability of the fractional Volterra integral-differential equation. Zadal et al. (2020) studied $\mathcal{UH}$ stability of impulsive integro-differential equations with RL boundary conditions. Dai (2020) studied a class of nonlinear FDE with integral boundary condition. Nowadays Ulam-Hyers stability has been one of the most active research topics in FDE.

The pantograph equation is one type of delay differential equations (DE). It is used in many different fields of pure and applied mathematics, physics, chemistry and other fields. The pantograph equation has been studied by many researchers and solved with numerical methods.

Vivek et co-worker. (2018) studied the existence and uniqueness results for nonlinear neutral pantograph equation with generalized fractional derivative:

$$\begin{cases} {}^{\rho}_{C}D^{\alpha}_{0+}u(t) = g(t, u(t), u(\mu t), {}^{\rho}_{C}D^{\alpha}_{0+}u(\mu t)), \quad t \in J = (0, T], \\ u(0) + h(u) = u_0, \end{cases}$$

where ${}^{\rho}_{C}D^{\alpha}_{0+}$ is Katugampola fractional derivative in Caputo sense, $\alpha \in \mathbb{R}^+$, $0 < \mu < 1$,$\rho > 0$ and $g : J \times X^3 \to X$, $h : C([0, T], X) \to X$ are continuous functions, by using Krasnoselskiis fixed point theorem.

Shah et al. (2018) studied existence of at least one solution and the $\mathcal{UH}, \mathcal{UHR}$ stabilities for fractional pantograph Differential Equation:

$$\begin{cases} {}^{C}D^{\alpha,\psi}u(t) = g(t, u(t), u(\mu t)), \quad t \in J = (0, T], \\ au(0) + bu(T) = c, \end{cases}$$

where ${}^{C}D^{\alpha,\psi}$ is the ψ Caputo fractional derivative of order α, $0 < \alpha < 1$ and $0 < \mu < 1$

and $g : J \times \mathbb{R}^2 \to \mathbb{R}$ a continuous function and a, b, c are constants, $a + b \neq 0$, by using Banach's fixed point theorem and Ulam-Hyrs stability theorem.

Vivek et al. (2018) studied the existence and stability of nonlinear neutral pantograph equations with Hilfer fractional derivative:

$$\begin{cases} D_{0^+}^{\alpha,\beta} u(t) = g(t, u(t), u(\lambda t), D_{0^+}^{\alpha,\beta} u(\mu t)), \quad t \in J = (0, T], \\ I^{1-\gamma} u(0) = \sum_{i=1}^{m} c_i u(\tau_i), \alpha < \gamma = \alpha + \beta - \alpha\beta < 1, \tau_i \in [a, T], \end{cases}$$

where $D_{0^+}^{\alpha,\beta}$ is the Hillfer FD, $0 < \alpha < 1$,$0 \leq \beta \leq 1$,$0 < \mu < 1$,$\rho > 0$ and let X be a Banach space $g : J \times X^3 \to X$, $g : C([0, T], X) \to X$ are continuous functions, by using Scheafers fixed point theorem and $\mathcal{GUH}$ stability.

Wongcharoen et al. (2020) studied boundary value problems involving the Hilfer fractional derivative:

$$\begin{cases} {}^{H}D^{\alpha,\beta} u(t) = g(t, u(t), u(\mu t)), \quad t \in (a, b], \\ u(a) = 0, \qquad Au(b) + BI^{\delta} u(\eta) = c, \qquad \eta \in (a, b), \end{cases}$$

where ${}^{H}D^{\alpha,\beta}$ is the Hilfer FD of order $\alpha, 1 < \alpha < 2$, β, $0 < \beta < 1$, $g : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a continuous function, I^{δ} is the RL fractional integral of order $\delta > 0$, $a > 0$, $A, B, c \in \mathbb{R}$ and $0 < \mu < 1$, by using Banach's fixed point theorem and Leray-Schauders nonlinear alternative.

Arshad et al. (2020) studied existence solution and the Ulam-Hyers stability for fractional pantograph differential equation:

$$\begin{cases} {}_{0}^{C}D^{\alpha} u(t) = g(t, u(t), u(\lambda t), {}_{0}^{C}D^{\alpha} u(t)), \quad t \in (0, T], 2 < \alpha \leq 3 \\ u(0) = -u(T), \qquad {}_{0}^{C}D^{p} u(0) = -{}_{0}^{C}D^{p} u(T), \qquad {}_{0}^{C}D^{q} u(0) = -{}_{0}^{C}D^{q} u(T) \end{cases}$$

where ${}_{0}^{C}D^{\alpha,\psi}$ is the Caputo fractional derivative of order α, $\lambda < 1$, $0 < p < 1$, $1 < q < 2$ and $g : J \times \mathbb{R}^3 \to \mathbb{R}$ a continuous, by using Banach's and Schauder's fixed point theorems and $\mathcal{UH}$, $\mathcal{GUH}$, $\mathcal{UHR}$, and $\mathcal{GUHR}$ stability.

The Langevin equation (LE) was presented by Langevin in 1908 to give an elaborate description of Brownian motion. In his work, Newton's second law was applied to a Brownian particle to invent the $F = ma$ of stochastic physics which is now called Langevin equation. The fractional Langevin equation is extensively studied in literature both from theoretical and numerical pesspectives. The fractional Langevin equation was proposed and studied by various researchers during recent years.

Anping & Chen (2017) studied a boundary value problem to LE involving two fractional orders:

$$\begin{cases} {}^C D^{\beta}\big({}^C D^{\alpha} + \lambda\big)u(t) = g(t, u(t)), \quad t \in (0, T], \\ u(0) = -u(T), \qquad u'(0) = u'(T) = 0, \end{cases}$$

where constant $T > 0$, $1 < \alpha \leq 2$, $0 < \beta \leq 1$, ${}^C D^{\beta}$ and ${}^C D^{\alpha}$ are the Caputo fractional derivatives, $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ a continuous function, and λ is a real number, by using Banach's fixed point theorem and Krasnoselskii's fixed point theorem.

Zhou & Qiao (2018) studied a class of fractional LE with integral and anti-periodic boundary conditions:

$$\begin{cases} {}^C D^{\beta}_{0+}\big({}^C D^{\alpha}_{0+} + \lambda\big)u(t) = g(t, u(t)), \quad 0 < t < 1, \\ u(0) = 0, \qquad u(1) = \mu \int_0^1 u(s)ds \qquad , {}^C D^{\beta}_{0+}u(0) + {}^C D^{\beta}_{0+}u(1) = 0, \end{cases}$$

where $0 < \alpha < 1$, $1 < \beta < 2$, $\lambda > 0$ and $\mu > 0$ are real numbers, ${}^C D^{\beta}_{0+}u(t)$ and ${}^C D^{\alpha}_{0+}u(t)$ are the Caputo fractional derivatives of order β and α, respectively and $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous function, by using Banach's fixed point theorem and Laray-Schauder's fixed point theorem.

Ahmad et al. (2019) presented the existence and uniqueness of solution for fractional LE involving RL as well as Caputo fractional derivatives and variable coefficient:

$$\begin{cases} {}^{RL}D^{q}\big({}^C D^{r} + \lambda(t)\big)u(t) = f(t, u(t)), \quad t \in (0, T], \\ u(\xi) = \alpha {}^C D^{\nu}u(\eta), \qquad u(T) = \beta I^{p}u(\zeta), \qquad 0 < \xi, \eta, \zeta < T, \end{cases}$$

where ${}^{RL}D^{q}$ denotes RL fractional derivative with order $q \in (0, 1)$, ${}^C D^{r}$, ${}^C D^{\nu}$ denotes Caputo fractional derivative of order r and ν, respectivly ,$0 < r < 1$, $0 < \nu < r$, I^{p} is the RL fractional integral of order $p > 0$, $\lambda \in C([0, T], \mathbb{R})$, $\alpha, \beta \in \mathbb{R}$ and $f : [0, 1] \times T \to \mathbb{R}$ is continuous function, by using Banach's fixed point theorem and Krassnoselskii's fixed point theorem.

Salem & Alghamdi (2019) studied the existence and uniqueness of solutions for the LE that has Caputo fractional derivatives of two different orders:

$$D^{\beta}\big(D^{\alpha}+\lambda\big)u(t)=g(t,u(t)),\quad t\in(0,1],$$

with the new boundary conditions

$$u(0)+u(1)=0,\quad {}^{C}D^{\alpha}u(0)=0,\quad {}^{C}D^{\alpha}u(1)=\sum_{i=1}^{m}\mu_i u(q_i),$$

where ${}^{C}D^{\alpha}$ and ${}^{C}D^{\beta}$ denotes the Caputo derivatives of generalized orders $\beta\in(1,2]$ and $\alpha\in(0,1]$, $\lambda\in\mathbb{R}$, $q_i\in(0,1)$, $i=1,2,3,...,m$ with $m\in\mathbb{N}$ and the function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous function.

Ahmad et al. (2019) developed the existence theorem for a nonlinear Langevin equation involving Caputo frational derivatives of different order and Riemann-Liouville fractional integral:

$$\begin{cases} {}^{C}D^{\alpha}\big({}^{C}D^{\beta}+\lambda\big)u(t)=g(t,u(t),I^{p}u(t)),\quad t\in(0,1]. \\ u(0)=\sum_{j=1}^{m}\nu_j u(\sigma_j),\qquad u'(0)=0,\qquad a_1u(1)+a_2u'(1)=\sum_{i=1}^{n}\rho_i\int_{\xi_i}^{\eta_i}u(s)ds \end{cases}$$

where ${}^{C}D^{\alpha}$,${}^{C}D^{\beta}$ are the Caputo fractional derivatives of orders α and β,$0<\alpha\leq 1$, $1<\beta\leq 2$ $p,\lambda>0$,$a_1,a_2,\nu_j,\rho_i\in\mathbb{R}$, $0<\sigma_j<\xi_i<\eta_i<1$, $i=1,2,...n$, $j=1,1,...m$, $m,n\in\mathbb{N}$ and $g:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ a continuous, by using Krasnoselskii's fixed point theorem and Leray-Schauder's nonlinear alternative.

Salem & Alghamdi (2020) discussed a boundary value problem for the nonlinear LE involving two distinct fractional derivatives of different orders:

$${}^{C}D^{p}\big({}^{C}D^{q}+\lambda\big)u(t)=g(t,u(t),{}^{C}D^{r}u(t)),\quad t\in(0,1].$$

with the multi-integral and multi-point boundary conditions:

$$u(0)=0,\quad D^{q}u(0)=0,\quad u(1)=\sum_{i=1}^{n}\alpha_i u(\eta_i)+\sum_{i=1}^{n}\beta_i\int_{0}^{\eta_i}u(s)ds.$$

where ${}^{C}D^{p}$,${}^{C}D^{q}$ and ${}^{C}D^{r}$ are the Liouville-Caputo fractional derivatives of orders $p\in(1,2]$,$q\in(0,1]$ and $0<r\leq q$, $\mu\in\mathbb{R}$ is the dissipative parameter, $\lambda,\alpha_i,\beta_i\in\mathbb{R}$,$\eta_i\in(0,1)$ $i=1,2,3,...,n$ such that $n\in\mathbb{N}$ with $w=\sum_{i=1}^{n}\alpha_i\eta_i^{q+1}\neq 1$ and the function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ a continuous function, by using Schauder's and Krasnoselskii-

Zabreiko's fixed point theorems.

Kosari et co-worker. (2020) studied the existence theorem for a non-linear LE with BVC:

$$\begin{cases} {}^{C}D_{0+}^{\beta}\left({}^{C}D_{0+}^{\alpha}+\lambda\right)u(t)=g(t,u(t),u'(t)), & t\in[0,1], \\ u(0)=u(1)=u'(0)=u'(1)=0, & 0<\alpha\leq 1, 1<\beta\leq 2, \end{cases}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivatives of orders α, λ is real number and $g : [0,1]\times\mathbb{R}^2\to\mathbb{R}$ a continuous, by using Banach's fixed point theorem and Schauder's fixed point theorem.

Sudsutad et al. (2020) study the existence and $\mathcal{UH}$ stability of LE with in the generalized proportional fractional derivatives.

$$\begin{cases} {}^{C}D^{\beta,\rho}\left({}^{C}D^{\alpha,\rho}+\lambda(t)\right)x(t)=f(t,x(t),{}^{C}D^{\alpha,\rho}x(t)), & t\in[0,1], \\ x(0)=\delta, \qquad x(1)=\kappa D^{\gamma,\rho}x(\eta)+\theta I^{\mu,\rho}x(\xi), \end{cases}$$

where ${}^{C}D^{\nu,\rho}$ and $D^{\nu,\rho}$ are the GPC and RL fractional derivatives of order $\nu\in\{\alpha,\beta,\gamma\}$, respectively, $0<\alpha,\beta,\gamma\leq 1, 1<\alpha+\beta\leq 2, \rho>0$, $I^{\mu,\rho}$ is the generalized proportional fractional integral of order $\mu>0$, $\rho>0$, $\lambda\in C([0,1],\mathbb{R})$, the nonlinear function $f\in C([0,1]\times\mathbb{R}^2,\mathbb{R})$, the given constants $\delta,\kappa,\theta\in\mathbb{R}$ and $\eta,\xi\in(0,1)$, by using Banach's fixed point theorem and Schauder's fixed point theorem.

This dissertation has focused on the two models of boundary value problems. The first is the fractional pantograph differential equation with mixed boundary condition of the form:

$$\begin{cases} {}_{a}^{C}D^{\alpha,\rho}u(t) = f_1(t, u(t), u(\lambda t), {}_{a}^{C}D^{\alpha,\rho}u(\lambda t)), \quad t \in (a, T], \quad 0 < \lambda < 1, \\ \displaystyle\sum_{i=1}^{m} \gamma_i u(\eta_i) + \sum_{j=1}^{n} \kappa_j {}_{a}^{C}D^{\beta_j,\rho}u(\xi_j) + \sum_{r=1}^{k} \sigma_r {}_{a}I^{\delta_r,\rho}u(\theta_r) = A, \end{cases} \tag{1.1}$$

where ${}_{a}^{C}D^{q,\rho}$ is the Caputo GPF derivative of order $q \in \{\alpha, \beta_j\}$ with $0 < \beta_j < \alpha \le 1$, for $j = 1, 2, \ldots, n$, $\rho > 0$, ${}_{a}I^{\delta_r,\rho}$ is the RL GPF integral of order $\delta_r > 0$ for $r = 1, 2, \ldots, k$, $0 < \rho \le 1$, the given constants $\gamma_i, \kappa_j, \sigma_r, A \in \mathbb{R}$, the points $\eta_i, \xi_j, \theta_r \in [a, T]$, $i = 1, 2, \ldots, m$, and $f_1 : [a, T] \times \mathbb{R}^3 \to \mathbb{R}$ is a given continuous function, $0 \le a < T$.

The second is the fractional Langevin integro-differential equation with nonlocal integral conditions of the form:

$$\begin{cases} {}_{a}^{C}D^{q_1,\rho}\big({}_{a}^{C}D^{q_2,\rho} + \lambda(t)\big)y(t) = f_2(t, y(t), y(\theta(t)), (\mathcal{K}y)(t)), \quad t \in [a, T], \\ \sum_{i=1}^{m} \kappa_i {}_{a}I^{\mu_i,\rho}y(\sigma_i) = \sum_{j=1}^{n} \alpha_j {}_{a}I^{\beta_j,\rho}y(\eta_j), \\ \sum_{k=1}^{p} \omega_k {}_{a}I^{\gamma_k,\rho}y(\psi_k) = \sum_{l=1}^{r} \nu_l {}_{a}I^{\varphi_l,\rho}y(\xi_l), \end{cases} \tag{1.2}$$

where ${}_{a}^{C}D^{q,\rho}$ denotes the Caputo type GPF derivatives of order q, $q \in \{q_1, q_2\}$ with $0 < q_1, q_2 \le 1$, $1 < q_1 + q_2 \le 2$, $0 < \rho \le 1$, ${}_{a}I^{w,\rho}$ is the GPF integral of order $w > 0$, $w \in \{\mu_i, \beta_j, \gamma_k, \varphi_l\}$, κ_i, α_j, ω_k, $\nu_l \in \mathbb{R}$, σ_i, η_j, ψ_k, $\xi_l \in (a, T)$, for all $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, p$, $l = 1, 2, \ldots, r$, m, n, p, $r \in \mathbb{N}$, $\lambda \in C(\mathbb{R}^+, \mathbb{R})$, $f_2 \in C([a, T] \times \mathbb{R}^3, \mathbb{R})$, $\theta \in C([a, T], [a, T])$ and

$$(\mathcal{K}y)(t) = \int_a^t \phi(t, s, y(s))ds, \quad t \in [a, T],$$

where $\phi \in C([a, T]^2 \times \mathbb{R}, [a, \infty))$.

Research objectives

(i) To prove the existence results of solution for fractional pantograph differential equations via generalized proportional fractional derivative (1.1).

(ii) To prove the $\mathcal{UH}$ stability, $\mathcal{GUH}$ stability, $\mathcal{UHR}$ stability and $\mathcal{GUHR}$ stability for fractional pantograph Differential equations via generalized proportional fractional derivative (1.1) .

(iii) To prove the existence results of solution for fractional Langevin differential equations via generalized proportional fractional derivative (1.2).

(iv) To prove the $\mathcal{UH}$ stability, $\mathcal{GUH}$ stability, $\mathcal{UHR}$ stability and $\mathcal{GUHR}$ stability for fractional Langevin differential equations via generalized proportional fractional derivative (1.2).

(v) Given some examples to support the main results for the problem (1.1) and (1.2).

Contribution to knowledge

(i) We obtain knowledge the existence and uniqueness of solution for the fractional pantograph differential equation (1.1) and the fractional Langevin integro-differential equation (1.2).

(ii) We obtain sufficient conditions for existence of solution for the fractional pantograph differential equation (1.1) and the fractional Langevin integro-differential equation (1.2).

(iii) We obtain the $\mathcal{UH}$ and $\mathcal{UHR}$ stability for the fractional pantograph differential equation (1.1) and the fractional Langevin integro-differential equation (1.2).

Scope of the study

In this dissertation, we will study the existence and uniqueness of the solution to fractional pantograph differential equation in problem (1.1) and fractional Langevin differential equation in problem (1.2) via the proportional fractional calculus by using Banach's fixed point theorem, Leray-Schauder's nonlinear alternative, Krasnoselskii's fixed point theorem and Schauder's fixed point theorem. We will present Ulam-Hyers stability results by using direct analysis methods. Different types of Ulam stability, such as $\mathcal{UH}$ stability, $\mathcal{GUH}$ stability, $\mathcal{UHR}$ stability, and $\mathcal{GUHR}$ stability, are obtained.

CHAPTER 2

BACKGROUND KNOWLEDGE

This chapter, we review about proportional fractional integrals and derivatives, Banach spaces, fixed point theorems and Ulam-Hyers stability.

Gamma function

Definition 2.1. The Gamma function is defined by

$$\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt.$$

From the definition, we have:

(i) $\Gamma(p+1) = p\Gamma(p)$,

(ii) $\Gamma(n+1) = n!$ for all $n \in \mathbb{Z}^+$,

(iii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$,

Beta function

Definition 2.2. The Beta function is determined by

$$\mathbb{B}(m,n) = \int_0^1 u^{m-1}(1-u)^{n-1} du.$$

From the definition, we have:

(i) $\mathbb{B}(m,n) = \mathbb{B}(n,m)$,

(ii) $\mathbb{B}(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$,

Proportional Fractional Calculus

Definition 2.3. (Jarad, Alqudah & Abdeljawad, 2020)

The generalized proportional fractional (GPF) integral of a function f of order $\delta > 0$

with $\rho \in (0,1]$ is given as

$$({}_aI^{\delta,\rho}f)(x) = \frac{1}{\rho^{\delta}\Gamma(\delta)}\int_a^x e^{\frac{\rho-1}{\rho}(x-\tau)}(x-\tau)^{\delta-1}f(\tau)d\tau,$$

Definition 2.4. (Jarad, Alqudah & Abdeljawad, 2020)

The Caputo type generalized proportional fractional derivative of a function f of order δ with $\rho \in (0,1]$ is given as

$$({}_a^CD^{\delta,\rho}f)(x) = \frac{1}{\rho^{n-\delta}\Gamma(n-\delta)}\int_a^x e^{\frac{\rho-1}{\rho}(x-\tau)}(x-\tau)^{n-\delta-1}D^{n,\rho}f(\tau)d\tau,$$

where $n = \lfloor\delta\rfloor + 1$, $\lfloor\delta\rfloor$ denotes the integer part of the real number δ.

Lemma 2.5. (Jarad, Alqudah & Abdeljawad, 2020)

For $n = \lfloor\delta\rfloor + 1$ and $\rho \in (0,1]$, we get

$$({}_a{}^CD^{\delta,\rho}\ {}_aI^{\delta,\rho}f(s))(x) = f(x),$$

and

$$({}_aI^{\delta,\rho}\ {}_a^CD^{\delta,\rho}f)(x) = f(x) - e^{\frac{\rho-1}{\rho}(x-a)}\sum_{k=0}^{n-1}\frac{D^{k,\rho}f(a)}{\rho^k k!}(x-a)^k.$$

Proposition 2.6. (Jarad, Alqudah & Abdeljawad, 2020)

Let $\delta \geq 0$, $\gamma > 0$, for all $\rho \in (0,1]$, $n = \lfloor\delta\rfloor + 1$, we get

(i) $\left({}_aI^{\delta,\rho}e^{\frac{\rho-1}{\rho}t}(\tau-a)^{\gamma-1}\right)(t) = \frac{\Gamma(\gamma)}{\Gamma(\delta+\gamma)\rho^{\delta}}e^{\frac{\rho-1}{\rho}t}(t-a)^{\delta+\gamma-1}, \quad \delta > 0$

(ii) $\left({}_a^CD^{\delta,\rho}e^{\frac{\rho-1}{\rho}t}(\tau-a)^{\gamma-1}\right)(t) = \frac{\rho^{\delta}\Gamma(\gamma)}{\Gamma(\gamma-\delta)}e^{\frac{\rho-1}{\rho}t}(t-a)^{\gamma-\delta-1}, \quad \gamma > n$

(iii) $\left({}_a^CD^{\delta,\rho}e^{\frac{\rho-1}{\rho}t}(\tau-a)^{k}\right)(t) = 0, \quad k = 0,1,\ldots,n-1.$

Theorem 2.7. (Jarad, Alqudah & Abdeljawad, 2020)

If $0 < \rho \leq 1$, $\delta > 0$, and $\gamma > 0$. Then, for continuous function f, we get

$${}_aI^{\delta,\rho}\left({}_aI^{\gamma,\rho}\right)f(t) = {}_aI^{\gamma,\rho}\left({}_aI^{\delta,\rho}\right)f(t) = {}_aI^{\delta+\gamma,\rho}f(t).$$

Theorem 2.8. (Jarad, Alqudah & Abdeljawad, 2020)
Let $0 \leq m < \lfloor \delta \rfloor + 1$ and f be integrable in $[a, t], t > a$. Then
$$D^{m,\rho}\left({}_aI^{\delta,\rho}\right)f(t) = {}_aI^{\delta-m,\rho}f(t).$$

Corollary 2.9. (Jarad, Alqudah & Abdeljaward, 2020)
Let $0 < \gamma < \delta$, $m - 1 < \gamma < m$. Then we get
$${}_aD^{\gamma,\rho}\left({}_aI^{\delta,\rho}\right)f(t) = {}_aI^{\delta-\gamma,\rho}f(t).$$

Mathematical Analysis and Banach space

This section, we give the definitions of metric spaces ($\mathcal{MS}$), vector spaces ($\mathcal{VS}$), normed spaces ($\mathcal{NS}$), Banach space ($\mathcal{BS}$) and properties of Banach spaces, functions in Banach spaces.

Definition 2.10. (Metric space)(Kreyszig, 1989)
Let a set X and a function $\rho : X \times X \to \mathbb{R}$, we say that (X, ρ) is a $\mathcal{MS}$ if and only if ρ satisfies the following properties:
for all $x, y, z \in X$

1. $\rho(x, y) \geq 0$ (d is a nonnegative real number),
2. $\rho(x, y) = 0$ iff $x = y$,
3. $\rho(x, y) = \rho(y, x)$ (symmetric),
4. $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ (triangle inequality),

Definition 2.11. (Vector space) (Kreyszig, 1989)

Let a non-empty set V,+ be a binary operator called addition of vectors, $\cdot$ be a binary operator called scalar multiplication of an element of $\mathbb{F}$ with a vector and $\mathbb{F}$ be a field. Then $(V, +, \cdot, \mathbb{F})$ is said to be a $\mathcal{VS}$ (linear space) if the following properties satisfy:

for all $u, v, w \in V$ and $k, m \in \mathbb{F}$

1. $u + v \in V$,
2. $u + v = v + u$;
3. $u + (v + w) = (u + v) + w$,
4. there is an element $\theta \in V$ such that $u + \theta = u$ for all $u \in V$,
5. for each $u \in V$, there is $-u \in V$ such that $u + (-u) = \theta$,
6. $k \cdot u \in V$,
7. $k \cdot (m \cdot u) = (k \cdot m) \cdot u$,
8. $1 \cdot u = u$, where 1 is the scalar $1 \in \mathbb{F}$;
9. $(k + m) \cdot u = k \cdot u + m \cdot u$,
10. $k \cdot (u + v) = k \cdot u + k \cdot v$.

Definition 2.12. (Normed space)(Kreyszig, 1989)

The $\mathcal{VS}$ over X is said to be $\mathcal{NS}$ if there is a function $\|.\| : X \to \mathbb{R}$ and, for all $x, y \in X$ and $\alpha \in \mathbb{R}$, the following are sastisfied:

1. $\|x\| \geq 0$ (i.e., $\|x\|$ is a nonnegative real number),
2. $\|x\| = 0$ if and only if $x = 0$,
3. $\|\alpha x\| = |\alpha| \|x\|$,
4. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

Definition 2.13. (Kreyszig, 1989)

Let X is a $\mathcal{NS}$, Then the open ball with radius $\delta > 0$ centered at a is given by

$$B_\delta(a) = \{x \in X : \|x - a\| < \delta\}.$$

Definition 2.14. (Kreyszig, 1989)

A subset S of a $\mathcal{NS}$ is said to be open if for all $x \in S$ there is $\delta > 0$ such that

$$B_\delta(x) \subset S.$$

Definition 2.15. (John & Nachtergaele, 2000)

A sequence $\{a_n\}$ is said to be convergent to a, if for any $\epsilon > 0$, there is an integer $N > 0$ such that

$$|a_n - a| < \epsilon,$$

for all $n > N$. We denote this by $\lim_{n\to\infty} a_n = a$.

Definition 2.16. (John & Nachtergaele, 2000)

Let $\{f_n\}$ be a sequence of functions. The sequence $\{f_n\}$ is said to be converge uniformly to f, if, given any $\epsilon > 0$, there exists a positive number N (with depends only upon ϵ) such that

$$|f_n(x) - f(x)| < \epsilon,$$

for all $n > N$ for every x. We denote this by $f_n \to^{unif} f$.

Definition 2.17. (John & Nachtergaele, 2000)

The sequence $\{a_n\}$ in the $\mathcal{NS}$ X is called a Cauchy sequence if and only if for each $\epsilon > 0$, there is an integer $N > 0$ such that

$$\|a_n - a_m\| < \epsilon, \qquad \text{for all} \qquad n, m \geq N.$$

Theorem 2.18. (John & Nachtergaele, 2000)

All convergent sequence in a $\mathcal{NS}$ is called a Cauchy sequence.

Definition 2.19. (John & Nachtergaele, 2000)

The $\mathcal{NS}$ X is said to be a $\mathcal{BS}$ if and only if all Cauchy sequence is convergent.

Definition 2.20. (Kreyszig, 1989)

Let X , Y be $\mathcal{VS}$ and S, U subsets of X, Y respectively. A mapping $T : S \to U$ is a rule which, given any $x \in S$, associates with it an element of U, denot by Tx or $T(x)$.

Definition 2.21. (John & Nachtergaele, 2000)

Subset S of a $\mathcal{NS}$ X is called compact if all sequence in S has a subsequence which converges to an element in S.

Definition 2.22. (John & Nachtergaele, 2000)

Subset S of a $\mathcal{NS}$ X is called relatively compact if all sequence in S has a subsequence which converges to an element in X.

Theorem 2.23. (John & Nachtergaele, 2000)

The set is said to be compact iff it closed and relatively compact.

Definition 2.24. (John & Nachtergaele, 2000)

Let a $\mathcal{NS}$ X. A family F of function on a set X is called uniformly bounded on X if there is $M > 0$ with $|f(x)| \leq M$ for all $x \in X$ and $f \in F$.

Definition 2.25. (John & Nachtergaele, 2000)

A family F of function on a normed space X is equicontinuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that, if $x, y \in X$ and $|x - y| < \delta$ then $|f(x) - f(y)| \leq \epsilon$ for all $f \in F$.

Definition 2.26. (Kreyszig, 1989)

Let U and V be $\mathcal{NS}$. A operator $T : U \to V$ is compact if, for all bounded sequence $\{u_n\}$ in U, the sequence $\{Tu_n\}$ in V contains a convergent subsequnce.

Definition 2.27. (Kreyszig, 1989)

Let U, V are Banach spaces, the map $T : D \subset U \to V$ is called completely continuous operator, if it is continuous and maps any bounded subset of D into a relatively compact subset of V.

Definition 2.28. (John & Nachtergaele, 2000)

A subset M of of a set X is called convex, if for all $x, y \in M$, there is a number $0 \leq \lambda \leq 1$, such that $\lambda x + (1 - \lambda)y \in M$.

Lemma 2.29. (Arzelá-Ascoli theorem) (John & Nachtergaele, 2000)
Let X be a $\mathcal{NS}$, and S a closed subset of X. Then S is relatively compact if and only if it is uniformly bounded and equicontinuous.

Definition 2.30. (John & Nachtergaele, 2000)
A map $T : U \to V$, where U, V are $\mathcal{NS}$. T is bounded if there exists constant $m > 0$ such that $\|Tx\| \leq m\|x\|$ for all $x \in X$.

Definition 2.31. (Kreyszig, 1989)
A mapping $T : S \to S$, where S is a subset of a $\mathcal{NS}$ X, is called a contraction mapping if there is a positive number $0 \leq L < 1$ such that

$$\|Tu - Tv\| \leq L\|u - v\| \qquad \text{for all} \qquad u, v \in S.$$

Some Fixed Points Theorems

In this section we give the statements of fixed point theorem, which are used to prove the results in this dessertation.

Definition 2.32. A point $x \in X$ is said to be a fixed point of the operator $T : X \to X$ if $Tx = x$.

Lemma 2.33. (Banach's fixed point theorem) (Xi Fu, 2013)
Let D be a closed subset of a $\mathcal{BS}$ E, and T be a contraction mapping from D to D. Then T has a unique fixed point.

Lemma 2.34. (Schaefer's fixed point theorem)(Shammakh & Alzumi, 2019)
Let $\mathbb{M}$ be a $\mathcal{BS}$, Assumme that $T : \mathbb{M} \to \mathbb{M}$ be a completely continuous operator and the set $D = \{x \in \mathbb{M} : x = \kappa Tx,\ 0 < \kappa \leq 1\}$ is bounded. Then T has a fixed point in $\mathbb{M}$.

Lemma 2.35. (Krasnoselskii's fixed point theorem) (Shammakh & Alzumi, 2019)
Let M be a closed, bounded, convex, and nonempty subset of a $\mathcal{BS}$. Let the operators A, B sastified :

(i) $Au + Bv \in M$ for all $u, v \in M$;

(ii) A is continuous and compact;

(iii) B be contraction mapping.

Then there is $x \in M$ such that $x = Ax + Bx$.

Lemma 2.36. (Leray-Schauder's Nonlinear alternative) (Xi Fu, 2013)
Let $\mathbb{M}$ be a $\mathcal{BS}$, C be a closed, convex subset of $\mathbb{M}$, X be an open subset of C, and $0 \in X$. Assume that $F : \overline{X} \to C$ is a continuous, compact (that is, $F(\overline{X})$ is a relatively compact subset of C) map. Then either

(i) F has a Fixed Point in $\overline{X}$, or

(ii) there exists $u \in \partial X$ (the boundary of X in C) and $\varrho \in (0, 1)$ with $u = \varrho F(u)$.

Ulam-Hyers stability

This section, we present the definitions of Ulam stability such as $\mathcal{UH}$ stable, $\mathcal{GUH}$ stable, $\mathcal{UHR}$ stable and $\mathcal{GUHR}$ stable.

Definition 2.37. (Sousa, Vanterler & Oliveira, 2018)
The problem (1.1) is called $\mathcal{UH}$ stable if there is a real number $\Phi > 0$ such that for each $\varrho > 0$ and for each solution $z \in \mathbb{E}^1 = C^1([a, T], \mathbb{R})$ of the inequality

$$\left| {}_a^C D^{\alpha,\rho} z(t) - f(t, z(t), z(\lambda t), ({}_a^C D^{\alpha,\rho} z)(\lambda t)) \right| \leq \varrho, \quad t \in [a, T], \tag{2.1}$$

there is a solution $u \in \mathbb{E}^1$ of the problem (1.1) such that

$$|z(t) - u(t)| \leq \Phi \varrho, \quad t \in [a, T]. \tag{2.2}$$

Definition 2.38. (Sousa, Vanterler & Oliveira, 2018)
The problem (1.1) is called $\mathcal{GUH}$ stable if there is a function $\Phi_f \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $\Phi_f(0) = 0$ such that, for each solution $z \in \mathbb{E}^1$ of inequality (2.1) there is a solution $u \in \mathbb{E}^1$ of the problem (1.1) such that

$$|z(t) - u(t)| \leq \Phi_f \varrho, \quad t \in [a, T]. \tag{2.3}$$

Definition 2.39. (Sousa, Vanterler & Oliveira, 2018)
The problem (1.1) is called $\mathcal{UHR}$ stable with respect to $\Phi_f \in C([a,T],\mathbb{R}^+)$ if there is a real number $C_{f,\Phi} > 0$ such that for each $\varrho > 0$ and for each solution $z \in \mathbb{E}^1$ of the inequality

$$\left|{}_a^C D^{\alpha,\rho} z(t) - f(t, z(t), z(\lambda t), ({}_a^C D^{\alpha,\rho} z)(\lambda t))\right| \leq \varrho \Phi_f(t), \quad t \in [a,T], \tag{2.4}$$

there is a solution $u \in \mathbb{E}^1$ of the problem (1.1) such that

$$|z(t) - u(t)| \leq C_{f,\Phi} \varrho \Phi_f(t), \quad t \in [a,T]. \tag{2.5}$$

Definition 2.40. (Sousa, Vanterler & Oliveira, 2018)
The problem (1.1) is called $\mathcal{GUHR}$ stable with respect to $\Phi_f \in C([a,T],\mathbb{R}^+)$ if there is a real number $C_{f,\Phi} > 0$ such that for each solution $z \in \mathbb{E}^1$ of the inequality

$$\left|{}_a^C D^{\alpha,\rho} z(t) - f(t, z(t), z(\lambda t), ({}_a^C D^{\alpha,\rho} z)(\lambda t))\right| \leq \Phi_f(t), \quad t \in [a,T], \tag{2.6}$$

there is a solution $u \in \mathbb{E}^1$ of the problem (1.1) such that

$$|z(t) - u(t)| \leq C_{f,\Phi} \Phi_f(t), \quad t \in [a,T]. \tag{2.7}$$

CHAPTER 3

EXISTENCE RESULTS

In this chapter, we will study the existence results for proportional Caputo fractional pantograph differential equation with mixed nonlocal boundary conditions of problem (1.1) and proportional fractional Langevin differential equation with nonlocal fractional integral conditions of problem (1.2), by using some fixed point theorems.

3.1 Existence Results for Proportional Fractional pantograph Differential Equation.

In this section we prove the existence results for proportional Caputo fractional pantograph differential equation with mixed non-local boundary conditions in problem (1.1), by using Banach's fixed point theorem, Leray-Schauder's nonlinear althenative and Krasnoselskii's fixed point theorem.

3.1.1 The solution of Linear Proportional Fractional pantograph Differential Equation with mixed Nonlocal Boundary Conditions

Lemma 3.1. *Let $h_1 : [a, T] \to \mathbb{R}$ a continuous function. Then, the function $u \in \mathbb{E}$ is a solution of the following problem with mixed nonlocal conditions of the form:*

$$\begin{cases} {}_a^C D^{\alpha,\rho} u(t) = h_1(t), \quad t \in (a, T], \\ \displaystyle\sum_{i=1}^{m} \gamma_i u(\eta_i) + \sum_{j=1}^{n} \kappa_j {}_a^C D^{\beta_j,\rho} u(\xi_j) + \sum_{r=1}^{k} \sigma_r {}_a I^{\delta_r,\rho} u(\theta_r) = A, \end{cases} \tag{3.1}$$

if and only if

$$\begin{aligned} u(t) &= {}_a I^{\alpha,\rho} h_1(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Delta} \Bigg(A - \sum_{i=1}^{m} \gamma_i {}_a I^{\alpha,\rho} h_1(\eta_i) \\ &\quad - \sum_{j=1}^{n} \kappa_j {}_a I^{\alpha-\beta_j,\rho} h_1(\xi_j) - \sum_{r=1}^{k} \sigma_r {}_a I^{\alpha+\delta_r,\rho} h_1(\theta_r) \Bigg), \end{aligned} \tag{3.2}$$

where

$$\Delta := \sum_{i=1}^{m} \gamma_i e^{\frac{\rho-1}{\rho}(\eta_i - a)} + \sum_{r=1}^{k} \frac{\sigma_r(\theta_r - a)^{\delta_r} e^{\frac{\rho-1}{\rho}(\theta_r - a)}}{\rho^{\delta_r}\Gamma(1+\delta_r)} \neq 0. \tag{3.3}$$

Proof. Assume that u is a solution of the problem (3.1), and using Lemma 2.5, we obtain

$$u(t) = {}_aI^{\alpha,\rho}h_1(t) + c_1 e^{\frac{\rho-1}{\rho}(t-a)}, \tag{3.4}$$

where arbitrary constants $c_1 \in \mathbb{R}$.

Taking the operators ${}_a^CD^{\beta_j,\rho}$ and ${}_aI^{\delta_r,\rho}$ into (3.4) with Proposition 2.6 (i), we obtain

$$\begin{aligned} {}_a^CD^{\beta_j,\rho}u(t) &= {}_aI^{\alpha-\beta_j,\rho}h_1(t), \\ {}_aI^{\delta_r,\rho}u(t) &= {}_aI^{\alpha+\delta_r,\rho}h_1(t) + c_1\frac{(t-a)^{\delta_r}e^{\frac{\rho-1}{\rho}(t-a)}}{\rho^{\delta_r}\Gamma(1+\delta_r)}. \end{aligned}$$

Using condition in (3.1), we have

$$\begin{aligned} A &= \sum_{i=1}^{m} \gamma_i {}_aI^{\alpha,\rho}h_1(\eta_i) + \sum_{j=1}^{n} \kappa_j {}_aI^{\alpha-\beta_j,\rho}h_1(\xi_j) + \sum_{r=1}^{k} \sigma_r {}_aI^{\alpha+\delta_r,\rho}h_1(\theta_r) \\ &\quad + c_1\left(\sum_{i=1}^{m} \gamma_i e^{\frac{\rho-1}{\rho}(\eta_i-a)} + \sum_{r=1}^{k} \frac{\sigma_r(\theta_r-a)^{\delta_r}e^{\frac{\rho-1}{\rho}(\theta_r-a)}}{\rho^{\delta_r}\Gamma(1+\delta_r)}\right). \end{aligned}$$

Solving the above equation, it follows that

$$c_1 = \frac{1}{\Delta}\left(A - \sum_{i=1}^{m} \gamma_i {}_aI^{\alpha,\rho}h_1(\eta_i) - \sum_{j=1}^{n} \kappa_j {}_aI^{\alpha-\beta_j,\rho}h_1(\xi_j) - \sum_{r=1}^{k} \sigma_r {}_aI^{\alpha+\delta_r,\rho}h_1(\theta_r)\right),$$

where Δ is given by (3.3). Substituting the value of c_1 in (3.4), we get (3.2).

Conversely, it is clearly to show by direct calculation that the function $u(t)$ is defined by (3.2) satisfies the problem (3.1). This completes the proof. □

For simplicity, we set

$$F_u(t) = f_1(t, u(t), u(\lambda t), F_u(\lambda t)).$$

Throughout this dissertation, the expression ${}_aI^{\alpha,\rho}F_u(s)(c)$ means that

$${}_aI^{q,\rho}F_u(s)(c) := \frac{1}{\rho^q\Gamma(q)}\int_a^c e^{\frac{\rho-1}{\rho}(c-s)}(c-s)^{q-1}F_u(s)ds,$$

In view of Lemma 3.1, we define an operator $\mathcal{K} : \mathbb{E} \to \mathbb{E}$ by

$$\begin{aligned}(\mathcal{K}u)(t) &= {}_aI^{\alpha,\rho}F_u(s)(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Delta}\Bigg(A - \sum_{i=1}^{m}\gamma_{ia}I^{\alpha,\rho}F_u(s)(\eta_i) \\ &\quad - \sum_{j=1}^{n}\kappa_{ja}I^{\alpha-\beta_j,\rho}F_u(s)(\xi_j) - \sum_{r=1}^{k}\sigma_{ra}I^{\alpha+\delta_r,\rho}F_u(s)(\theta_r)\Bigg), \end{aligned} \tag{3.5}$$

It should be noticed that the problem (1.1) has solution if and only if the operator $\mathcal{K}$ has fixed point.

To proceed further, we introduce the following hypotheses:

(A_1) Assume that $f_1 : [a,T] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function.

(A_2) There is a positive constants $\mathbb{L}_1, \mathbb{L}_2$ such that

$$|f_1(t, u_1, v_1, w_1) - f_1(t, u_2, v_2, w_2)| \leq \mathbb{L}_1(|u_1 - u_2| + |v_1 - v_2|) + \mathbb{L}_2|w_1 - w_2|$$

for all u_i, v_i, $w_i \in \mathbb{R}$, $i = 1, 2$ and $t \in [a,T]$.

(A_3) there is a continuous nondecreasing function $\psi : [0,\infty) \to [0,\infty)$, $p \in C([a,T], \mathbb{R}^+)$ and $q \in C([a,T], \mathbb{R}^+ \cup \{0\})$ such that

$$|f_1(t, u(t), u(\lambda t), {}_a^CD^{\alpha,\rho}u(\lambda t))| \leq p(t)\psi(|u(t)|) + q(t)\left|{}_a^CD^{\alpha,\rho}u(\lambda t)\right|,$$

for all $t \in [a,T]$, $x \in \mathbb{R}$;

where $p_0 = \sup_{t\in[a,T]}\{p(t)\}$, $q_0 = \sup_{t\in[a,T]}\{q(t)\}$ and $q_0 < 1$.

(A_4) there exists a positive constant N_1 such that

$$\frac{N_1}{\frac{|A|}{|\Delta|} + p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(N_1)\Lambda} > 1,$$

where Λ is defined by (3.7).

(A_5) $|f_1(t, u, v, w)| \leq g(t), \forall (t, u, v, w) \in [a,T] \times \mathbb{R}^3$ and $g \in C([a,T], \mathbb{R}^+)$.

3.1.2 Existence and uniqueness result via Banach's Fixed Point Theorem

Theorem 3.2. Assume that the hypotheses (A_1) and (A_2) are satisfiesd.
If

$$\frac{2\mathbb{L}_1\Lambda}{1-\mathbb{L}_2} < 1, \tag{3.6}$$

then the problem (1.1) has a unique solution $(u \in \mathbb{E})$ on $[a, T]$, where

$$\begin{aligned} \Lambda \ &= \ \frac{(T-a)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{1}{|\Delta|}\Bigg(\sum_{i=1}^{m} \frac{|\gamma_i|(\eta_i - a)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} \\ &\quad + \sum_{j=1}^{n} \frac{|\kappa_j|(\xi_j - a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)} + \sum_{r=1}^{k} \frac{|\sigma_r|(\theta_r - a)^{\alpha+\delta_r}}{\rho^{\alpha+\delta_r}\Gamma(\alpha+\delta_r+1)}\Bigg). \end{aligned} \tag{3.7}$$

Proof. Let $\sup_{t\in[a,T]} |f_1(t,0,0,0)| := M_1 < \infty$. Next, we set $B_{r_1} := \{u \in \mathbb{E} : \|u\| \leq r_1\}$ with

$$r_1 \geq \frac{M_1\Lambda|\Delta| + |A|(1-\mathbb{L}_2)}{|\Delta|[1-(2\mathbb{L}_1\Lambda+\mathbb{L}_2)]}, \qquad 2\mathbb{L}_1\Lambda + \mathbb{L}_2 < 1, \tag{3.8}$$

where Δ and Λ are given by (3.3) and (3.7), respectively. Observe that B_{r_1} is bounded, closed, and convex subset of $\mathbb{E}$. The proof is divided into two steps:

Step I. To show that $\mathcal{K}B_{r_1} \subset B_{r_1}$.

For any $u \in B_{r_1}$, we obtain

$$\begin{aligned} |(\mathcal{K}u)(t)| \ &\leq \ {}_aI^{\alpha,\rho}|F_u(s)|(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{|\Delta|}\Bigg(|A| + \sum_{i=1}^{m} |\gamma_i|_aI^{\alpha,\rho}|F_u(s)|(\eta_i) \\ &\quad + \sum_{j=1}^{n} |\kappa_j|_aI^{\alpha-\beta_j,\rho}|F_u(s)|(\xi_j) + \sum_{r=1}^{k} |\sigma_r|_aI^{\alpha+\delta_r,\rho}|F_u(s)|(\theta_r)\Bigg). \end{aligned}$$

From hypotheses (A_2), we get

$$\begin{aligned} |F_u(t)| \ &\leq \ |f_1(t,u(t),u(\lambda t),F_u(\lambda t)) - f_1(t,0,0,0)| + |f_1(t,0,0,0)| \\ &\leq \ 2\mathbb{L}_1|u(t)| + \mathbb{L}_2|F_u(t)| + M_1 \\ &\leq \ \frac{2\mathbb{L}_1|u(t)| + M_1}{1-\mathbb{L}_2}. \end{aligned}$$

This implies that

$$
\begin{aligned}
|(\mathcal{K}u)(t)| \leq\ & {}_aI^{\alpha,\rho}\left(\frac{2\mathbb{L}_1|u(s)|+M_1}{1-\mathbb{L}_2}\right)(t) \\
& +\frac{e^{\frac{\rho-1}{\rho}(t-a)}}{|\Delta|}\Bigg(|A|+\sum_{i=1}^{m}|\gamma_i|{}_aI^{\alpha,\rho}\left(\frac{2\mathbb{L}_1|u(s)|+M_1}{1-\mathbb{L}_2}\right)(\eta_i) \\
& +\sum_{j=1}^{n}|\kappa_j|{}_aI^{\alpha-\beta_j,\rho}\left(\frac{2\mathbb{L}_1|u(s)|+M_1}{1-\mathbb{L}_2}\right)(\xi_j) \\
& +\sum_{r=1}^{k}|\sigma_r|{}_aI^{\alpha+\delta_r,\rho}\left(\frac{2\mathbb{L}_1|u(s)|+M_1}{1-\mathbb{L}_2}\right)(\theta_r)\Bigg).
\end{aligned}
$$

By using $0 < e^{\frac{\rho-1}{\rho}(u-s)} \leq 1$ for any $a \leq s < u \leq T$, we obtain

$$
\begin{aligned}
|(\mathcal{K}u)(t)| &\leq \left(\frac{2\mathbb{L}_1 r_1+M_1}{1-\mathbb{L}_2}\right)\Bigg(\frac{(T-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}+\frac{1}{|\Delta|}\Bigg(\sum_{i=1}^{m}\frac{|\gamma_i|(\eta_i-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} \\
&\quad +\sum_{j=1}^{n}\frac{|\kappa_j|(\xi_j-a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)}+\sum_{r=1}^{k}\frac{|\sigma_r|(\theta_r-a)^{\alpha+\delta_r}}{\rho^{\alpha+\delta_r}\Gamma(\alpha+\delta_r+1)}\Bigg)\Bigg)+\frac{|A|}{|\Delta|} \\
&= \left(\frac{2\mathbb{L}_1 r_1+M_1}{1-\mathbb{L}_2}\right)\Lambda+\frac{|A|}{|\Delta|}\leq r_1,
\end{aligned}
$$

which implies that $\|\mathcal{K}u\| \leq r_1$. Therefore, $\mathcal{K}B_{r_1} \subset B_{r_1}$.

Step II. To represent that the operator $\mathcal{K} : \mathbb{E} \to \mathbb{E}$ is a contraction mapping.

For any u, $v \in \mathbb{E}$, for all $t \in [a, T]$, we get

$$
\begin{aligned}
& |(\mathcal{K}u)(t)-(\mathcal{K}v)(t)| \\
\leq\ & {}_aI^{\alpha,\rho}|F_u(s)-F_v(s)|(T)+\frac{e^{\frac{\rho-1}{\rho}(T-a)}}{|\Delta|}\Bigg(\sum_{i=1}^{m}|\gamma_i|{}_aI^{\alpha,\rho}|F_u(s)-F_v(s)|(\eta_i) \\
& +\sum_{j=1}^{n}|\kappa_j|{}_aI^{\alpha-\beta_j,\rho}|F_u(s)-F_v(s)|(\xi_j) \\
& +\sum_{r=1}^{k}|\sigma_r|{}_aI^{\alpha+\delta_r,\rho}|F_u(s)-F_v(s)|(\theta_r)\Bigg), \qquad (3.9)
\end{aligned}
$$

and

$$\begin{aligned}|F_u(t)-F_v(t)| &\leq |f_1(t,u(t),u(\lambda t),F_u(\lambda t))-f_1(t,v(t),v(\lambda t),F_v(\lambda t))|\\ &\leq \mathbb{L}_1(|u(t)-v(t)|+|u(\lambda t)-v(\lambda t)|)+\mathbb{L}_2|F_u(\lambda t)-F_v(\lambda t)|\\ &\leq 2\mathbb{L}_1|u(t)-v(t)|+\mathbb{L}_2|F_u(t)-F_v(t)|\\ &\leq \frac{2\mathbb{L}_1}{1-\mathbb{L}_2}|u(t)-v(t)|. \qquad (3.10)\end{aligned}$$

Then, by substituting (3.10) in (3.9), we get

$$\begin{aligned}&|(\mathcal{K}u)(t)-(\mathcal{K}v)(t)|\\ \leq\ & {}_aI^{\alpha,\rho}\left(\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}|u(s)-v(s)|\right)(T)\\ &+\frac{e^{\frac{\rho-1}{\rho}(T-a)}}{|\Delta|}\left(\sum_{i=1}^{m}|\gamma_i|{}_aI^{\alpha,\rho}\left(\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}|u(s)-v(s)|\right)(\eta_i)\right.\\ &+\sum_{j=1}^{n}|\kappa_j|{}_aI^{\alpha-\beta_j,\rho}\left(\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}|u(s)-v(s)|\right)(\xi_j)\\ &\left.+\sum_{r=1}^{k}|\sigma_r|{}_aI^{\alpha+\delta_r,\rho}\left(\frac{2\mathbb{L}_1}{1-\mathbb{L}_2}|u(s)-v(s)|\right)(\theta_r)\right)\\ \leq\ & \frac{2\mathbb{L}_1}{1-\mathbb{L}_2}\left[\frac{(T-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}+\frac{1}{|\Delta|}\left(\sum_{i=1}^{m}\frac{|\gamma_i|(\eta_i-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right.\right.\\ &\left.\left.+\sum_{j=1}^{n}\frac{|\kappa_j|(\xi_j-a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)}+\sum_{r=1}^{k}\frac{|\sigma_r|(\theta_r-a)^{\alpha+\delta_r}}{\rho^{\alpha+\delta_r}\Gamma(\alpha+\delta_r+1)}\right)\right]\|u-v\|\\ =\ & \frac{2\mathbb{L}_1\Lambda}{1-\mathbb{L}_2}\|u-v\|,\end{aligned}$$

this implies that $\|\mathcal{K}u-\mathcal{K}v\|\leq(2\mathbb{L}_1\Lambda)/(1-\mathbb{L}_2)\|u-v\|$. As $(2\mathbb{L}_1\Lambda)/(1-\mathbb{L}_2)<1$, hence, the operator $\mathcal{K}$ is a contraction mapping. Thus, by the Banach's fixed point theorem (Lemma 2.33), the problem (1.1) has a unique solution in $\mathbb{E}$. The proof is completed. □

3.1.3 Existence Result via Leray-Schauder's Nonlinear Alternative

Theorem 3.3. Suppose that the hypotheses (A_3) and (A_4) are satisfied. Therefore the problem (1.1) has at least one solution on $[a,T]$.

Proof. Let the operator $\mathcal{K}$ given by (3.5). First of all, we show that $\mathcal{K}$ maps bounded sets into bounded sets in $\mathbb{E}$. For a constant $r_2>0$, let $B_{r_2} := \{u \in \mathbb{E} : \|u\| \leq r_2\}$, for all $t \in [a,T]$, we have

$$\begin{aligned} |(\mathcal{K}u)(t)| &\leq {}_aI^{\alpha,\rho}|F_u(s)|(T) + \frac{e^{\frac{\rho-1}{\rho}(T-a)}}{|\Delta|}\Bigg(|A| + \sum_{i=1}^{m}|\gamma_i|_aI^{\alpha,\rho}|F_u(s)|(\eta_i) \\ &\quad + \sum_{j=1}^{n}|\kappa_j|_aI^{\alpha-\beta_j,\rho}|F_u(s)|(\xi_j) + \sum_{r=1}^{k}|\sigma_r|_aI^{\alpha+\delta_r,\rho}|F_u(s)|(\theta_r)\Bigg). \end{aligned}$$

It follows from (A_3) that

$$\begin{aligned} |{}_a^CD^{\alpha,\rho}u(t)| &= |F_u(s)| \\ &\leq p(t)\psi(|u(t)|) + q(t)|{}_a^CD^{\alpha,\rho}u(\lambda t)| \\ &\leq p(t)\psi(|u(t)|) + q(t)|{}_a^CD^{\alpha,\rho}u(t)|. \end{aligned}$$

This implies that

$$\left|{}_a^CD^{\alpha,\rho}u(t)\right| \leq \frac{p(t)\psi(|u(t)|)}{1-q(t)}.$$

By using $0 < e^{\frac{\rho-1}{\rho}(u-s)} \leq 1$, for any $a \leq s < u \leq T$, we get

$$\begin{aligned} |(\mathcal{K}u)(t)| &\leq \frac{|A|}{|\Delta|} + p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|u\|)\Bigg[\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_a^T(T-s)^{\alpha-1}ds \\ &\quad + \frac{1}{|\Delta|}\Bigg(\sum_{i=1}^{m}\frac{|\gamma_i|}{\rho^\alpha\Gamma(\alpha)}\int_a^{\eta_i}(\eta_i-s)^{\alpha-1}ds \\ &\quad + \sum_{j=1}^{n}\frac{|\kappa_j|}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j)}\int_a^{\xi_j}(\xi_j-s)^{\alpha-\beta_j-1}ds \\ &\quad + \sum_{r=1}^{k}\frac{|\sigma_r|}{\rho^{\alpha+\delta_r}\Gamma(\alpha+\delta_r)}\int_a^{\theta_r}(\theta_r-s)^{\alpha+\delta_r-1}ds\Bigg)\Bigg] \\ &= \frac{|A|}{|\Omega|} + p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|u\|)\Lambda, \end{aligned}$$

which leads to

$$\|\mathcal{K}u\| \leq \frac{|A|}{|\Delta|} + p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(r_2)\Lambda := D_1.$$

Next, we will show that the operator $\mathcal{K}$ maps from bounded sets into equicontinuous sets of $\mathbb{E}$. Let points $\tau_1, \tau_2 \in [a, T]$, and $\tau_1 < \tau_2$ and $u \in B_{r_2}$. Then we obtain

$$
\begin{aligned}
& |(\mathcal{K}u)(\tau_2) - (\mathcal{K}u)(\tau_1)| \\
\leq\ & |{}_aI^{\alpha,\rho}F_u(s)(\tau_2) - {}_aI^{\alpha,\rho}F_u(s)(\tau_1)| \\
& + \frac{\left|e^{\frac{\rho-1}{\rho}(\tau_2-a)} - e^{\frac{\rho-1}{\rho}(\tau_1-a)}\right|}{|\Delta|}\Bigg(|A| + \sum_{i=1}^{m}|\gamma_i|{}_aI^{\alpha,\rho}|F_u(s)|(\eta_i) \\
& + \sum_{j=1}^{n}|\kappa_j|{}_aI^{\alpha-\beta_j,\rho}|F_u(s)|(\xi_j) + \sum_{r=1}^{k}|\sigma_r|{}_aI^{\alpha+\delta_r,\rho}|F_u(s)|(\theta_r)\Bigg) \\
\leq\ & \frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_a^{\tau_2} e^{\frac{\rho-1}{\rho}(\tau_2-s)}(\tau_2-s)^{\alpha-1}F_u(s)ds - \int_a^{\tau_1} e^{\frac{\rho-1}{\rho}(\tau_1-s)}(\tau_1-s)^{\alpha-1}F_u(s)ds\right| \\
& + \frac{\left|e^{\frac{\rho-1}{\rho}(\tau_2-a)} - e^{\frac{\rho-1}{\rho}(\tau_1-a)}\right|}{|\Delta|}\Bigg(|A| + \sum_{i=1}^{m}|\gamma_i|{}_aI^{\alpha,\rho}|F_u(s)|(\eta_i) \\
& + \sum_{j=1}^{n}|\kappa_j|{}_aI^{\alpha-\beta_j,\rho}|F_u(s)|(\xi_j) + \sum_{r=1}^{k}|\sigma_r|{}_aI^{\alpha+\delta_r,\rho}|F_u(s)|(\theta_r)\Bigg) \\
\leq\ & \frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_a^{\tau_1} e^{\frac{\rho-1}{\rho}(\tau_2-s)}(\tau_2-s)^{\alpha-1}F_u(s)ds - \int_a^{\tau_1} e^{\frac{\rho-1}{\rho}(\tau_1-s)}(\tau_1-s)^{\alpha-1}F_u(s)ds\right| \\
& + \frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_{\tau_1}^{\tau_2} e^{\frac{\rho-1}{\rho}(\tau_2-s)}(\tau_2-s)^{\alpha-1}F_u(s)ds\right| \\
& + \frac{\left|e^{\frac{\rho-1}{\rho}(\tau_2-a)} - e^{\frac{\rho-1}{\rho}(\tau_1-a)}\right|}{|\Delta|}\Bigg(|A| + \sum_{i=1}^{m}|\gamma_i|{}_aI^{\alpha,\rho}|F_u(s)|(\eta_i) \\
& + \sum_{j=1}^{n}|\kappa_j|{}_aI^{\alpha-\beta_j,\rho}|F_u(s)|(\xi_j) + \sum_{r=1}^{k}|\sigma_r|{}_aI^{\alpha+\delta_r,\rho}|F_u(s)|(\theta_r)\Bigg) \\
\leq\ & p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|u\|) \\
& \times\Bigg(\frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_a^{\tau_1}\left(e^{\frac{\rho-1}{\rho}(\tau_2-s)}(\tau_2-s)^{\alpha-1} - e^{\frac{\rho-1}{\rho}(\tau_1-s)}(\tau_1-s)^{\alpha-1}\right)ds\right| \\
& + \frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_{\tau_1}^{\tau_2} e^{\frac{\rho-1}{\rho}(\tau_2-s)}(\tau_2-s)^{\alpha-1}ds\right|\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|u\|)\frac{\left|e^{\frac{\rho-1}{\rho}(\tau_2-a)}-e^{\frac{\rho-1}{\rho}(\tau_1-a)}\right|}{|\Delta|}\\
&\times\left(|A|+\sum_{i=1}^{m}|\gamma_i|_aI^{\alpha,\rho}(\eta_i)+\sum_{j=1}^{n}|\kappa_j|_aI^{\alpha-\beta_j,\rho}(\xi_j)+\sum_{r=1}^{k}|\sigma_r|_aI^{\alpha+\delta_r,\rho}(\theta_r)\right)\\
\leq\ & p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|u\|)\\
&\times\left(\frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_a^{\tau_1}e^{\frac{\rho-1}{\rho}(\tau_1-s)}\left(e^{\frac{\rho-1}{\rho}(\tau_2-\tau_1)}(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right)ds\right|\right.\\
&\left.+\frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_{\tau_1}^{\tau_2}e^{\frac{\rho-1}{\rho}(\tau_2-s)}(\tau_2-s)^{\alpha-1}ds\right|\right)\\
&+\frac{p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|u\|)}{|\Delta|}\left|e^{\frac{\rho-1}{\rho}(\tau_2-a)}-e^{\frac{\rho-1}{\rho}(\tau_1-a)}\right|\left[|A|+\left(\sum_{i=1}^{m}\frac{|\gamma_i|(\eta_i-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right.\right.\\
&\left.\left.+\sum_{j=1}^{n}\frac{|\kappa_j|(\xi_j-a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)}+\sum_{r=1}^{k}\frac{|\sigma_r|(\theta_r-a)^{\alpha+\delta_r}}{\rho^{\alpha-\delta_r}\Gamma(\alpha+\delta_r+1)}\right)\right].
\end{aligned}
$$

For all $s,\tau_1,\tau_2\in[a,T]$, such that $\tau_2>\tau_1$, $a\leq s\leq\tau_1$ and $a\leq s\leq\tau_2$, we have $0<e^{\frac{\rho-1}{\rho}(\tau_2-\tau_1)}\leq 1$, $0<e^{\frac{\rho-1}{\rho}(\tau_1-s)}\leq 1$ and $0<e^{\frac{\rho-1}{\rho}(\tau_2-s)}\leq 1$.

Thus

$$
\begin{aligned}
&|(\mathcal{K}u)(\tau_2)-(\mathcal{K}u)(\tau_1)|\\
\leq\ & p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|u\|)\\
&\times\left(\frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_a^{\tau_1}\left((\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right)ds\right|\right.\\
&\left.+\frac{1}{\rho^\alpha\Gamma(\alpha)}\left|\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}ds\right|\right)\\
&+\frac{p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|u\|)}{|\Delta|}\left|e^{\frac{\rho-1}{\rho}(\tau_2-a)}-e^{\frac{\rho-1}{\rho}(\tau_1-a)}\right|\left[|A|+\left(\sum_{i=1}^{m}\frac{|\gamma_i|(\eta_i-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\right.\right.\\
&\left.\left.+\sum_{j=1}^{n}\frac{|\kappa_j|(\xi_j-a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)}+\sum_{r=1}^{k}\frac{|\sigma_r|(\theta_r-a)^{\alpha+\delta_r}}{\rho^{\alpha-\delta_r}\Gamma(\alpha+\delta_r+1)}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|u\|)\\
&\times\frac{1}{\rho^{\alpha}\Gamma(\alpha+1)}\Big(\,|(\tau_2-a)^{\alpha}-(\tau_1-a)^{\alpha}-(\tau_2-\tau_1)^{\alpha}|+(\tau_2-\tau_1)^{\alpha}\Big)\\
&+\frac{p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|u\|)}{|\Delta|}\left|e^{\frac{\rho-1}{\rho}(\tau_2-a)}-e^{\frac{\rho-1}{\rho}(\tau_1-a)}\right|\Bigg[|A|+\Bigg(\sum_{i=1}^{m}\frac{|\gamma_i|(\eta_i-a)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\\
&+\sum_{j=1}^{n}\frac{|\kappa_j|(\xi_j-a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)}+\sum_{r=1}^{k}\frac{|\sigma_r|(\theta_r-a)^{\alpha+\delta_r}}{\rho^{\alpha-\delta_r}\Gamma(\alpha+\delta_r+1)}\Bigg)\Bigg]
\end{aligned}
$$

Clearly, by independent of $u\in B_{r_2}$ and the inequality,we have $|(\mathcal{K}u)(\tau_2)-(\mathcal{K}u)(t\tau_1)|\to 0$ as $\tau_2\to\tau_1$. Thus, from the Arzelá-Ascoli theorem the $\mathcal{K}:\mathbb{E}\to\mathbb{E}$ is completely continuous.

Latterly, we shall show that there is an open set $\mathcal{X}\subseteq\mathbb{E}$ with $u\neq\varrho\mathcal{K}(u)$ for $\varrho\in(0,1)$ and $u\in\partial\mathcal{X}$.

Let $u\in\mathbb{E}$ be a solution of $u=\varrho\mathcal{K}u$ for $\varrho\in[0,1]$. Then, for $t\in[a,T]$, we obtain

$$
\begin{aligned}
|u(t)| &= |\varrho(\mathcal{K}u)(t)|\\
&\le \frac{|A|}{|\Delta|}+p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|u\|)\Bigg[\frac{(T-s)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}+\frac{1}{|\Delta|}\Bigg(\sum_{i=1}^{m}\frac{|\gamma_i|(\eta_i-a)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\\
&\quad+\sum_{j=1}^{n}\frac{|\kappa_j|(\xi_j-a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)}+\sum_{r=1}^{k}\frac{|\sigma_r|(\theta_r-a)^{\alpha+\delta_r}}{\rho^{\alpha+\delta_r}\Gamma(\alpha+\delta_r+1)}\Bigg)\Bigg]\\
&= \frac{|A|}{|\Delta|}+p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|u\|)\Lambda,
\end{aligned}
$$

Taking the norm with $t\in[a,T]$, get that

$$\|u\|\le\frac{|A|}{|\Delta|}+p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|u\|)\Lambda.$$

Consequently, we get

$$\frac{\|u\|}{\frac{|A|}{|\Delta|}+p_0\left(\frac{2-q_0}{1-q_0}\right)\psi(\|u\|)\Lambda}\le 1.$$

In view of (A_4), there exists N_1 such that $\|u\|\neq N_1$. Let us set

$$\mathcal{X}=\{u\in\mathbb{E}:\|u\|<N_1\}\qquad\text{and}\qquad\mathcal{Y}=\mathcal{X}\cap B_{r_2}.$$

Note that the operator $\mathcal{K}:\overline{\mathcal{Y}}\to\mathbb{E}$ is continuous and completely continuous. From the choice of $\mathcal{Y}$, there is no $u\in\partial\mathcal{Y}$ such that $u=\varrho\mathcal{K}u$ for some $\varrho\in(0,1)$. Hence, by the nonlinear alternative of Leray-Schauder type (Lemma 2.36), we deduce that $\mathcal{K}$ has fixed point $u \in \overline{\mathcal{Y}}$ which implies that the problem (1.1) has at least one solution on $[a, T]$. This completes the proof. □

3.1.4 Existence Result via Krasnoselskii's Fixed Point Theorem

Theorem 3.4. Suppose that the hypotheses (A_1), (A_2) and (A_5) are satisfiesd. Then the problem (1.1) has at least one solution on $[a, T]$ provided

$$\frac{2\mathbb{L}_1}{(1-\mathbb{L}_2)|\Delta|}\left(\sum_{i=1}^{m}\frac{|\gamma_i|(\eta_i-a)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}+\sum_{j=1}^{n}\frac{|\kappa_j|(\xi_j-a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)}\right.$$
$$\left.+\sum_{r=1}^{k}\frac{|\sigma_r|(\theta_r-a)^{\alpha+\delta_r}}{\rho^{\alpha+\delta_r}\Gamma(\alpha+\delta_r+1)}\right)<1. \tag{3.11}$$

Proof. Setting $\sup_{t\in[a,T]}|g(t)| = \|g\|$ and choosing

$$r_3 \geq \frac{|A|}{|\Delta|} + \|g\|\Lambda, \tag{3.12}$$

we consider $B_{r_3} := \{u \in \mathbb{E} : \|u\| \leq r_3\}$. We determine the operators $\mathcal{K}_1$ and $\mathcal{K}_2$ on B_{r_3} by

$$\begin{aligned}
(\mathcal{K}_1u)(t) &= {}_aI^{\alpha,\rho}F_u(s)(t), \quad t \in [a, T],\\
(\mathcal{K}_2u)(t) &= \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Delta}\left(A-\sum_{i=1}^{m}\gamma_{i\,a}I^{\alpha,\rho}F_u(s)(\eta_i)-\sum_{j=1}^{n}\kappa_{j\,a}I^{\alpha-\beta_j,\rho}F_u(s)(\xi_j)\right.\\
&\quad\left.-\sum_{r=1}^{k}\sigma_{r\,a}I^{\alpha+\delta_r,\rho}F_u(s)(\theta_r)\right),
\end{aligned}$$

for $t \in [a,T]$. Note that $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$. For any $u, v \in B_{r_3}$, we obtain

$$\begin{aligned}
\|\mathcal{K}_1 u + \mathcal{K}_2 v\| &\leq \sup_{t\in[a,t]} \Bigg\{ {}_aI^{\alpha,\rho}|F_u(s)|(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{|\Delta|}\Bigg(|A| + \sum_{i=1}^{m} |\gamma_i| {}_aI^{\alpha,\rho}|F_v(s)|(\eta_i) \\
&\quad + \sum_{j=1}^{n} |\kappa_j| {}_aI^{\alpha-\beta_j,\rho}|F_v(s)|(\xi_j) + \sum_{r=1}^{k} |\sigma_r| {}_aI^{\alpha+\delta_r,\rho}|F_v(s)|(\theta_r) \Bigg) \Bigg\} \\
&\leq \frac{|A|}{|\Delta|} + \|g\| \Bigg\{ \frac{(T-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{1}{|\Delta|}\Bigg(\sum_{i=1}^{m} \frac{|\gamma_i|(\eta_i - a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} \\
&\quad + \sum_{j=1}^{n} \frac{|\kappa_j|(\xi_j - a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)} + \sum_{r=1}^{k} \frac{|\sigma_r|(\theta_r - a)^{\alpha+\delta_r}}{\rho^{\alpha+\delta_r}\Gamma(\alpha+\delta_r+1)} \Bigg) \Bigg\} \\
&= \frac{|A|}{|\Delta|} + \|g\|\Lambda \leq r_3.
\end{aligned}$$

This implies that $\mathcal{K}_1 u + \mathcal{K}_2 v \in B_{r_3}$ which satisfies assumption (i) of Lemma 2.35 It is easy to see, using (3.11), that the operator $\mathcal{K}_2$ is a contraction mapping and also assumption (iii) of Lemma 2.35 holds.

To show that assumption (ii) of Lemma 2.35 is satisfied. Let u_n be a sequence such that $u_n \to u$ in $\mathbb{E}$ where $n \to \infty$. Then for each $t \in [a,T]$, we get

$$\begin{aligned}
|(\mathcal{K}_1 u_n)(t) - (\mathcal{K}_1 u)(t)| &\leq {}_aI^{\alpha,\rho}|F_{u_n}(s) - F_u(s)|(t) \\
&\leq \frac{(T-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\|F_{u_n} - F_u\|
\end{aligned}$$

Since f is continuous, F_u is also continuous. Therefore, we obtain

$$\|\mathcal{K}_1 u_n - \mathcal{K}_1 u\| \to 0, \quad \text{as} \quad n \to \infty.$$

Thus, this shows that the operator $\mathcal{K}_1 u$ is continuous. Also, the set $\mathcal{K}_1 B_{r_3}$ is uniformly bounded as

$$\|\mathcal{K}_1 u\| \leq \frac{(T-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)}\|g\|.$$

Next, we shall show the compactness of the operator $\mathcal{K}_1$.

Let $\sup_{(t,u,v,w)\in[a,t]\times\mathbb{R}^3} |f(t,u,v,w)| = f^* < \infty$, for any $\tau_1, \tau_2 \in [a,T]$

with $\tau_1 \leq \tau_2$, we have

$$
\begin{aligned}
& |(\mathcal{K}_1 u)(\tau_2) - (\mathcal{K}_1 u)(\tau_1)| \\
= \;& |{}_aI^{\alpha,\rho} F_u(s)(\tau_2) - {}_aI^{\alpha,\rho} F_u(s)(\tau_1)| \\
= \;& \frac{1}{\rho^\alpha \Gamma(\alpha)} \left| \int_a^{\tau_2} e^{\frac{\rho-1}{\rho}(\tau_2 - s)} \big(\tau_2 - s\big)^{\alpha-1} F_u(s) ds - \int_a^{\tau_1} e^{\frac{\rho-1}{\rho}(\tau_1 - s)} \big(\tau_1 - s\big)^{\alpha-1} F_u(s) ds \right| \\
\leq \;& \frac{1}{\rho^\alpha \Gamma(\alpha)} \left| \int_a^{\tau_1} \left(e^{\frac{\rho-1}{\rho}(\tau_2 - s)} \big(\tau_2 - s\big)^{\alpha-1} - e^{\frac{\rho-1}{\rho}(\tau_1 - s)} \big(\tau_1 - s\big)^{\alpha-1} \right) ds \right| |F_u(s)| \\
& + \frac{1}{\rho^\alpha \Gamma(\alpha)} \int_{\tau_1}^{\tau_2} e^{\frac{\rho-1}{\rho}(\tau_2 - s)} \big(\tau_2 - s\big)^{\alpha-1} |F_u(s)| ds \\
\leq \;& \frac{f^*}{\rho^\alpha \Gamma(\alpha)} \left(\left| \int_a^{\tau_1} e^{\frac{\rho-1}{\rho}(\tau_1 - s)} \left(e^{\frac{\rho-1}{\rho}(\tau_2 - \tau_1)} \big(\tau_2 - s\big)^{\alpha-1} - \big(\tau_1 - s\big)^{\alpha-1} \right) ds \right| \right. \\
& \left. + \int_{\tau_1}^{\tau_2} e^{\frac{\rho-1}{\rho}(\tau_2 - s)} \big(\tau_2 - s\big)^{\alpha-1} ds \right)
\end{aligned}
$$

For all $s, \tau_1, \tau_2 \in [a, T]$, such thaat $\tau_2 > \tau_1$, $a \leq s \leq \tau_1$ and $a \leq s \leq \tau_2$, we have $0 < e^{\frac{\rho-1}{\rho}(\tau_2 - \tau_1)} \leq 1$, $0 < e^{\frac{\rho-1}{\rho}(\tau_1 - s)} \leq 1$ and $0 < e^{\frac{\rho-1}{\rho}(\tau_2 - s)} \leq 1$.

Thus

$$
\begin{aligned}
& |(\mathcal{K}_1 u)(\tau_2) - (\mathcal{K}_1 u)(\tau_1)| \\
\leq \;& \frac{f^*}{\rho^\alpha \Gamma(\alpha)} \left(\left| \int_a^{\tau_1} \left(\big(\tau_2 - s\big)^{\alpha-1} - \big(\tau_1 - s\big)^{\alpha-1} \right) ds \right| + \int_{\tau_1}^{\tau_2} \big(\tau_2 - s\big)^{\alpha-1} ds \right) \\
\leq \;& \frac{f^*}{\rho^\alpha \Gamma(\alpha+1)} \left(\left| \big(\tau_2 - a\big)^\alpha - \big(\tau_1 - a\big)^\alpha - \big(\tau_2 - \tau_1\big)^\alpha \right| + \big(\tau_2 - \tau_1\big)^\alpha \right),
\end{aligned}
$$

which is independent of u and $|(\mathcal{K}_1 u)(\tau_2) - (\mathcal{K}_1 u)(\tau_1)| \to 0$ as $\tau_2 \to \tau_1$. Thus, the set $\mathcal{K}_1 B_{r_3}$ is equi-continuous and the operator $\mathcal{K}_1$ maps bounded subsets into relatively compact subsets. It follows that the set $\mathcal{K}_1 B_{r_3}$ is relatively compact. Then, by the Arzelá-Ascoli theorem, the operator $\mathcal{K}_1$ is compact on B_{r_3}. Thus all the assumptions of Lemma 2.35 are satisfied. So, the conclusion of Lemma 2.35 implies that the problem (1.1) has at least one solution on $[a, T]$. The proof is completed. □

3.2 Existence Results for Proportional Fractional Langevin Differential Equation.

In this section we prove the existence results of the proportional fractional Langevin differential equation with nonlocal fractional integral conditions in problem (1.2).

3.2.1 The solution of Linear Proportional Fractional Langevin Differential Equation

Lemma 3.5. *Let $h_2 : [a,T] \to \mathbb{R}$ is a continuous function. Then, the function $y \in \mathbb{E}$ is the solution to the following linear generalized proportional fractional Langevin equation equipped with nonlocal fractional integral conditions:*

$$\begin{cases} {}_a^CD^{q_1,\rho}\big({}_a^CD^{q_2,\rho} + \lambda(t)\big)y(t) = h_2(t), \quad t \in [a,T], \\ \displaystyle\sum_{i=1}^{m} \kappa_{i\,a}I^{\mu_i,\rho}y(\sigma_i) = \sum_{j=1}^{n} \alpha_{j\,a}I^{\beta_j,\rho}y(\eta_j), \\ \displaystyle\sum_{k=1}^{p} \omega_{k\,a}I^{\gamma_k,\rho}y(\psi_k) = \sum_{l=1}^{r} \nu_{l\,a}I^{\varphi_l,\rho}y(\xi_l), \end{cases} \tag{3.13}$$

if and only if

$$\begin{aligned} y(t) \;=\;& {}_aI^{q_1+q_2,\rho}h_2(t) - {}_aI^{q_2,\rho}\lambda(t)y(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right) \\ & \times\left(\sum_{j=1}^{n}\alpha_j\left[{}_aI^{q_1+q_2+\beta_j,\rho}h_2(\eta_j) - {}_aI^{q_2+\beta_j,\rho}\lambda(\eta_j)y(\eta_j)\right]\right. \\ & \left.-\sum_{i=1}^{m}\kappa_i\left[{}_aI^{q_1+q_2+\mu_i,\rho}h_2(\sigma_i) - {}_aI^{q_2+\mu_i,\rho}\lambda(\sigma_i)y(\sigma_i)\right]\right) \\ & +\left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\left(\sum_{l=1}^{r}\nu_l\big[{}_aI^{q_1+q_2+\varphi_l,\rho}h_2(\xi_l)\right. \\ & -{}_aI^{q_2+\varphi_l,\rho}\lambda(\xi_l)y(\xi_l)\big] \\ & \left.-\sum_{k=1}^{p}\omega_k\left[{}_aI^{q_1+q_2+\gamma_k,\rho}h_2(\psi_k) - {}_aI^{q_2+\gamma_k,\rho}\lambda(\psi_k)y(\psi_k)\right]\right)\Bigg], \end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
\Omega_1 &= \sum_{i=1}^{m}\frac{\kappa_i(\sigma_i-a)^{q_2+\mu_i}e^{\frac{\rho-1}{\rho}(\sigma_i-a)}}{\rho^{q_2+\mu_i}\Gamma(q_2+\mu_i+1)}-\sum_{j=1}^{n}\frac{\alpha_j(\eta_j-a)^{q_2+\beta_j}e^{\frac{\rho-1}{\rho}(\eta_j-a)}}{\rho^{q_2+\beta_j}\Gamma(q_2+\beta_j+1)}, &(3.15)\\
\Omega_2 &= \sum_{i=1}^{m}\frac{\kappa_i(\sigma_i-a)^{\mu_i}e^{\frac{\rho-1}{\rho}(\sigma_i-a)}}{\rho^{\mu_i}\Gamma(\mu_i+1)}-\sum_{j=1}^{n}\frac{\alpha_j(\eta_j-a)^{\beta_j}e^{\frac{\rho-1}{\rho}(\eta_j-a)}}{\rho^{\beta_j}\Gamma(\beta_j+1)}, &(3.16)\\
\Omega_3 &= \sum_{k=1}^{p}\frac{\omega_k(\psi_k-a)^{q_2+\gamma_k}e^{\frac{\rho-1}{\rho}(\psi_k-a)}}{\rho^{q_2+\gamma_k}\Gamma(q_2+\gamma_k+1)}-\sum_{l=1}^{r}\frac{\nu_l(\xi_l-a)^{q_2+\varphi_l}e^{\frac{\rho-1}{\rho}(\xi_l-a)}}{\rho^{q_2+\varphi_l}\Gamma(q_2+\varphi_l+1)}, &(3.17)\\
\Omega_4 &= \sum_{k=1}^{p}\frac{\omega_k(\psi_k-a)_k^{\gamma}e^{\frac{\rho-1}{\rho}(\psi_k-a)}}{\rho^{\gamma_k}\Gamma(\gamma_k+1)}-\sum_{l=1}^{r}\frac{\nu_l(\xi_l-a)^{\varphi_l}e^{\frac{\rho-1}{\rho}(\xi_l-a)}}{\rho^{\varphi_l}\Gamma(\varphi_l+1)}., &(3.18)\\
\Omega &= \Omega_1\Omega_4-\Omega_2\Omega_3\neq 0. &(3.19)
\end{aligned}$$

Proof. Assume that y is a solution of (3.13), and using Lemma 2.5 with Proposition 2.6 (i), we obtain

$$y(t)={}_aI^{q_1+q_2,\rho}h_2(t)-{}_aI^{q_2,\rho}\lambda(t)y(t)+c_1\frac{(t-a)^{q_2}e^{\frac{\rho-1}{\rho}(t-a)}}{\rho^{q_2}\Gamma(q_2+1)}+c_2e^{\frac{\rho-1}{\rho}(t-a)}, \quad (3.20)$$

where c_1, $c_2\in\mathbb{R}$ are arbitrary.

Taking the GPF integral operator ${}_aI^{w,\rho}$ into (3.20), we obtain

$$\begin{aligned}
{}_aI^{w,\rho}y(t) &= {}_aI^{q_1+q_2+w,\rho}h_2(t)-{}_aI^{q_2+w,\rho}\lambda(t)y(t)\\
&\quad +c_1\frac{(t-a)^{q_2+w}e^{\frac{\rho-1}{\rho}(t-a)}}{\rho^{q_2+w}\Gamma(q_2+w+1)}+c_2\frac{(t-a)^{w}e^{\frac{\rho-1}{\rho}(t-a)}}{\rho^{w}\Gamma(w+1)}. \qquad (3.21)
\end{aligned}$$

Substituting $w\in\{\mu_i,\beta_j,\gamma_k,\varphi_l\}$, $t\in\{\sigma_i,\eta_j,\psi_k,\xi_l\}$ in (3.21), respectively, and applying the boundary conditions of the problem (3.13), we have

$$\begin{aligned}
\Omega_1c_1+\Omega_2c_2 &= \sum_{j=1}^{n}\alpha_j\left[{}_aI^{q_1+q_2+\beta_j,\rho}h_2(\eta_j)-{}_aI^{q_2+\beta_j,\rho}\lambda(\eta_j)y(\eta_j)\right]\\
&\quad -\sum_{i=1}^{m}\kappa_i\left[{}_aI^{q_1+q_2+\mu_i,\rho}h_2(\sigma_i)-{}_aI^{q_2+\mu_i,\rho}\lambda(\sigma_i)y(\sigma_i)\right],\\
\Omega_3c_1+\Omega_4c_2 &= \sum_{l=1}^{r}\nu_l\left[{}_aI^{q_1+q_2+\varphi_l,\rho}h_2(\xi_l)-{}_aI^{q_2+\varphi_l,\rho}\lambda(\xi_l)y(\xi_l)\right]\\
&\quad -\sum_{k=1}^{p}\omega_k\left[{}_aI^{q_1+q_2+\gamma_k,\rho}h_2(\psi_k)-{}_aI^{q_2+\gamma_k,\rho}\lambda(\psi_k)y(\psi_k)\right].
\end{aligned}$$

Solving the above system for c_1 and c_2, we have

$$\begin{aligned} c_1 &= \frac{1}{\Omega}\Bigg[\Omega_4\Bigg(\sum_{j=1}^{n}\alpha_j\left[{}_aI^{q_1+q_2+\beta_j,\rho}h_2(\eta_j) - {}_aI^{q_2+\beta_j,\rho}\lambda(\eta_j)y(\eta_j)\right] \\ &\quad -\sum_{i=1}^{m}\kappa_i\left[{}_aI^{q_1+q_2+\mu_i,\rho}h_2(\sigma_i) - {}_aI^{q_2+\mu_i,\rho}\lambda(\sigma_i)y(\sigma_i)\right]\Bigg) \\ &\quad -\Omega_2\Bigg(\sum_{l=1}^{r}\nu_l\left[{}_aI^{q_1+q_2+\varphi_l,\rho}h_2(\xi_l) - {}_aI^{q_2+\varphi_l,\rho}\lambda(\xi_l)y(\xi_l)\right] \\ &\quad -\sum_{k=1}^{p}\omega_k\left[{}_aI^{q_1+q_2+\gamma_k,\rho}h_2(\psi_k) - {}_aI^{q_2+\gamma_k,\rho}\lambda(\psi_k)y(\psi_k)\right]\Bigg)\Bigg], \\ c_2 &= \frac{1}{\Omega}\Bigg[\Omega_1\Bigg(\sum_{l=1}^{r}\nu_l\left[{}_aI^{q_1+q_2+\varphi_l,\rho}h_2(\xi_l) - {}_aI^{q_2+\varphi_l,\rho}\lambda(\xi_l)y(\xi_l)\right] \\ &\quad -\sum_{k=1}^{p}\omega_k\left[{}_aI^{q_1+q_2+\gamma_k,\rho}h_2(\psi_k) - {}_aI^{q_2+\gamma_k,\rho}\lambda(\psi_k)y(\psi_k)\right]\Bigg) \\ &\quad -\Omega_3\Bigg(\sum_{j=1}^{n}\alpha_j\left[{}_aI^{q_1+q_2+\beta_j,\rho}h_2(\eta_j) - {}_aI^{q_2+\beta_j,\rho}\lambda(\eta_j)y(\eta_j)\right] \\ &\quad -\sum_{i=1}^{m}\kappa_i\left[{}_aI^{q_1+q_2+\mu_i,\rho}h_2(\sigma_i) - {}_aI^{q_2+\mu_i,\rho}\lambda(\sigma_i)y(\sigma_i)\right]\Bigg)\Bigg], \end{aligned}$$

where Ω_1, Ω_2, Ω_3, Ω_4 and Ω are defined by (3.15)–(3.19), respectively. Substituting the values c_1 and c_2 into (3.20), we get (3.14).

Conversely, it is clearly to shown by direct calculation that the function $y(t)$ defined by (3.14) satisfies the problem (3.13) under the given boundary conditions. The proof is completed. □

Next, we establish the existence results of solutions for the problem (1.2). Fixed Point Theorems are employed to prove the results.

For simplicity, we let

$$F_y(t) = f_2(t, y(t), y(\theta(t)), (\mathcal{K}y)(t)).$$

Throughout this paper, the expression ${}_aI^{b,\rho}F_y(s)(c)$

means that

$${}_aI^{b,\rho}F_y(s)(c) = \frac{1}{\rho^b\Gamma(b)}\int_a^c e^{\frac{\rho-1}{\rho}(c-s)}(c-s)^{b-1}F_y(s)ds.$$

In view of Lemma 3.5, we define an operator $\mathcal{Q} : \mathbb{E} \to \mathbb{E}$ by

$$\begin{aligned}(\mathcal{Q}y)(t) &= {}_aI^{q_1+q_2,\rho}F_y(s)(t) - {}_aI^{q_2,\rho}\lambda(s)y(s)(t) \\ &+\frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right) \\ &\times\left(\sum_{j=1}^{n}\alpha_j\left[{}_aI^{q_1+q_2+\beta_j,\rho}F_y(s)(\eta_j) - {}_aI^{q_2+\beta_j,\rho}\lambda(s)y(s)(\eta_j)\right]\right. \\ &\left.-\sum_{i=1}^{m}\kappa_i\left[{}_aI^{q_1+q_2+\mu_i,\rho}F_y(s)(\sigma_i) - {}_aI^{q_2+\mu_i,\rho}\lambda(s)y(s)(\sigma_i)\right]\right) \\ &+\left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right) \\ &\times\left(\sum_{l=1}^{r}\nu_l\left[{}_aI^{q_1+q_2+\varphi_l,\rho}F_y(s)(\xi_l) - {}_aI^{q_2+\varphi_l,\rho}\lambda(s)y(s)(\xi_l)\right]\right. \\ &\left.-\sum_{k=1}^{p}\omega_k\left[{}_aI^{q_1+q_2+\gamma_k,\rho}F_y(s)(\psi_k) - {}_aI^{q_2+\gamma_k,\rho}\lambda(s)y(s)(\psi_k)\right]\right)\Bigg] \quad (3.22)\end{aligned}$$

To proceed further, we introduce the following hypotheses:

(H_1) Let $f_2 : [a,T] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function.

(H_2) Let $\lambda : [a,T] \to \mathbb{R}$ be a continuous function.

(H_3) There exist positive constants L_1, L_2 such that

$$|f_2(t,x_1,y_1,z_1) - f_2(t,x_2,y_2,z_2)| \le L_1\left(|x_1-x_2| + |y_1-y_2|\right) + L_2|z_1-z_2|$$

for any $x_j, y_j, z_j \in \mathbb{R}$, $j = 1,2$ and $t \in [a,T]$.

(H_4) Let $\phi : [a,T]^2 \times \mathbb{R} \to \mathbb{R}$ be a continuous function and there is a constant $\phi_0 > 0$ such that

$$|\phi(t,s,x) - \phi(t,s,y)| \le \phi_0|x-y|$$

for each $(t,s) \in [a,T]^2$ and $x, y \in \mathbb{R}$.

(H_5) $|f_2(t,x,y,z)| \le g(t), \forall(t,x,y,z) \in [a,T] \times \mathbb{R}^3$ and $g \in C([a,T],\mathbb{R}^+)$.

(H_6) There are non-negative continuous functions h_1, h_2, h_3, $h_4 \in \mathbb{E}$, such that

$$|f_2(t,x,y,z)| \le h_1(t) + h_2(t)|x| + h_3(t)|y| + h_4(t)|z|,$$

$$\forall(x,y,z) \in \mathbb{R}, \quad t \in [a,T],$$

with $h_1^* = \sup_{t\in[a,T]} h_1(t)$, $h_2^* = \sup_{t\in[a,T]} h_2(t)$, $h_3^* = \sup_{t\in[a,T]} h_3(t)$, $h_4^* = \sup_{t\in[a,T]} h_4(t)$.

For the sake of computational convenience, we make use of the following constants:

$$\Lambda_1 = \frac{1}{|\Omega|}\left(\frac{|\Omega_4|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} + |\Omega_3|\right), \quad (3.23)$$

$$\Lambda_2 = \frac{1}{|\Omega|}\left(\frac{|\Omega_2|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} + |\Omega_1|\right), \quad (3.24)$$

$$\begin{aligned}\Lambda_3(u) &= \frac{(T-a)^u}{\rho^{q_1+q_2}\Gamma(u+1)} \\ &+\Lambda_1\left(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{u+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(u+\beta_j+1)} + \sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{u+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(u+\mu_i+1)}\right) \\ &+\Lambda_2\left(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{u+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(u+\varphi_l+1)} + \sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{u+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(u+\gamma_k+1)}\right), \quad (3.25)\end{aligned}$$

$$\begin{aligned}\Lambda_4 &= {}_aI^{q_2,\rho}|\lambda(s)|(T) \\ &+\Lambda_1\left(\sum_{j=1}^{n}|\alpha_j|{}_aI^{q_2+\beta_j,\rho}|\lambda(s)|(\eta_j) + \sum_{i=1}^{m}|\kappa_i|{}_aI^{q_2+\mu_i,\rho}|\lambda(s)|(\sigma_i)\right) \\ &+\Lambda_2\left(\sum_{l=1}^{r}|\nu_l|{}_aI^{q_2+\varphi_l,\rho}|\lambda(s)|(\xi_l) + \sum_{k=1}^{p}|\omega_k|{}_aI^{q_2+\gamma_k,\rho}|\lambda(s)|(\psi_k)\right). \quad (3.26)\end{aligned}$$

where $u \in \{q_1+q_2, q_1+q_2+1\}$.

3.2.2 Existence and uniqueness result via Banach's Fixed Point Theorem

Theorem 3.6. Assume that the hypotheses (H_1), (H_2), (H_3) and (H_4) are satisfied. If

$$2L_1\Lambda_3(q_1+q_2) + L_2\phi_0\Lambda_3(q_1+q_2+1) + \Lambda_4 < 1 \quad (3.27)$$

and Λ_1, Λ_2, $\Lambda_3(u)$, $u \in \{q_1+q_2, q_1+q_2+1\}$ and Λ_4 are given by (3.23), (3.24), (3.25) and (3.26), respectively, then, problem (1.2) has a unique solution in the space $\mathbb{E}$.

Proof. Let $\sup_{t\in[a,T]}|f_2(t,0,0,0)| := M_1 < \infty$. Next, we set $B_{R_1} := \{y \in \mathbb{E} : \|y\| \le R_1\}$ with

$$R_1 \ge \frac{M_1\Lambda_3(q_1+q_2)}{1-[2L_1\Lambda_3(q_1+q_2) - L_2\phi_0\Lambda_3(q_1+q_2+1) - \Lambda_4]}, \quad (3.28)$$

where $\Lambda_3(u)$, $u \in \{q_1+q_2, q_1+q_2+1\}$ and Λ_4 are defined by (3.25) and (3.26), respectively. Note that B_{R_1} is bounded, closed, and convex subset of $\mathbb{E}$. The proof is divided into two steps:

Step I. To show that $\mathcal{Q}B_{R_1} \subset B_{R_1}$.

For any $y \in B_{R_1}$, we have

$$
\begin{aligned}
&|(\mathcal{Q}y)(t)| \\
\leq\ & {}_aI^{q_1+q_2,\rho}|F_y(s)|(t) + {}_aI^{q_2,\rho}|\lambda(s)||y(s)|(t) \\
&+\frac{e^{\frac{\rho-1}{\rho}(t-a)}}{|\Omega|}\Bigg[\left(\frac{|\Omega_4|(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_3|\right) \\
&\times\left(\sum_{j=1}^{n}|\alpha_j|\Big[{}_aI^{q_1+q_2+\beta_j,\rho}|F_y(s)|(\eta_j)+{}_aI^{q_2+\beta_j,\rho}|\lambda(s)||y(s)|(\eta_j)\Big]\right. \\
&\left.+\sum_{i=1}^{m}|\kappa_i|\Big[{}_aI^{q_1+q_2+\mu_i,\rho}|F_y(s)|(\sigma_i)+{}_aI^{q_2+\mu_i,\rho}|\lambda(s)||y(s)|(\sigma_i)\Big]\right) \\
&+\left(|\Omega_1|+\frac{|\Omega_2|(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right) \\
&\times\left(\sum_{l=1}^{r}|\nu_l|\Big[{}_aI^{q_1+q_2+\varphi_l,\rho}|F_y(s)|(\xi_l)+{}_aI^{q_2+\varphi_l,\rho}|\lambda(s)||y(s)|(\xi_l)\Big]\right. \\
&\left.+\sum_{k=1}^{p}|\omega_k|\ \big[{}_aI^{q_1+q_2+\gamma_k,\rho}|F_y(s)|(\psi_k)+{}_aI^{q_2+\gamma_k,\rho}|\lambda(s)||y(s)|(\psi_k)\big]\right)\Bigg] \\
\leq\ & {}_aI^{q_1+q_2,\rho}(|F_y(s)-f_2(s,0,0,0)|+|f_2(s,0,0,0)|)(T)+{}_aI^{q_2,\rho}|\lambda(s)||y(s)|(T) \\
&+\frac{e^{\frac{\rho-1}{\rho}(T-a)}}{|\Omega|}\times\Bigg[\left(\frac{|\Omega_4|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_3|\right)\left(\sum_{j=1}^{n}|\alpha_j|\Big[{}_aI^{q_1+q_2+\beta_j,\rho}(|F_y(s)\right. \\
&-f_2(s,0,0,0)|+|f_2(s,0,0,0)|)(\eta_j)+{}_aI^{q_2+\beta_j,\rho}|\lambda(s)||y(s)|(\eta_j)\Big] \\
&+\sum_{i=1}^{m}|\kappa_i|\Big[{}_aI^{q_1+q_2+\mu_i,\rho}(|F_y(s)-f_2(s,0,0,0)|+|f_2(s,0,0,0)|)(\sigma_i) \\
&\left.+{}_aI^{q_2+\mu_i,\rho}|\lambda(s)||y(s)|(\sigma_i)\Big]\right)+\left(|\Omega_1|+\frac{|\Omega_2|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right) \\
&\times\left(\sum_{l=1}^{r}|\nu_l|\Bigg[{}_aI^{q_1+q_2+\varphi_l,\rho}(|F_y(s)-f_2(s,0,0,0)|+|f_2(s,0,0,0)|)(\xi_l)\right. \\
&+{}_aI^{q_2+\varphi_l,\rho}|\lambda(s)||y(s)|(\xi_l)\Bigg]
\end{aligned}
$$

$$+\sum_{k=1}^{p}|\omega_k|\Bigg[{}_aI^{q_1+q_2+\gamma_k,\rho}(|F_y(s)-f_2(s,0,0,0)|+|f_2(s,0,0,0)|)(\psi_k)$$
$$+{}_aI^{q_2+\gamma_k,\rho}|\lambda(s)||y(s)|(\psi_k)\Bigg]\Bigg)\Bigg].$$

By using the property $0<e^{\frac{\rho-1}{\rho}(u-s)}\leq 1$ for $a\leq s<u\leq T$ and (H_3)-(H_4), we obtain

$$\begin{aligned}
&|(\mathcal{Q}y)(t)|\\
\leq\ & {}_aI^{q_1+q_2,\rho}(L_1(|y(s)|+|y(\theta(s))|)+L_2|(\mathcal{K}y)(s))|+M_1)(T)\\
&+{}_aI^{q_2,\rho}|\lambda(s)||y(s)|(T)+\frac{e^{\frac{\rho-1}{\rho}(T-a)}}{|\Omega|}\Bigg[\left(\frac{|\Omega_4|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_3|\right)\\
&\times\Bigg(\sum_{j=1}^{n}|\alpha_j|\Big[{}_aI^{q_1+q_2+\beta_j,\rho}(L_1(|y(s)|+|y(\theta(s))|)+L_2|(\mathcal{K}y)(s))|\\
&+M_1)(\eta_j)+{}_aI^{q_2+\beta_j,\rho}|\lambda(s)||y(s)|(\eta_j)\Big]\\
&+\sum_{i=1}^{m}|\kappa_i|\Big[{}_aI^{q_1+q_2+\mu_i,\rho}(L_1(|y(s)|+|y(\theta(s))|)+L_2|(\mathcal{K}y)(s))|\\
&+M_1)(\sigma_i)+{}_aI^{q_2+\mu_i,\rho}|\lambda(s)||y(s)|(\sigma_i)\Big]\Bigg)\\
&+\left(|\Omega_1|+\frac{|\Omega_2|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\Bigg(\sum_{l=1}^{r}|\nu_l|\Big[{}_aI^{q_1+q_2+\varphi_l,\rho}(L_1(|y(s)|\\
&+|y(\theta(s))|)+L_2|(\mathcal{K}y)(s))|+M_1)(\xi_l)+{}_aI^{q_2+\varphi_l,\rho}|\lambda(s)||y(s)|(\xi_l)\Big]\\
&+\sum_{k=1}^{p}|\omega_k|\Big[{}_aI^{q_1+q_2+\gamma_k,\rho}(L_1(|y(s)|\\
&+|y(\theta(s))|)+L_2|(\mathcal{K}y)(s))|+M_1)(\psi_k)+{}_aI^{q_2+\gamma_k,\rho}|\lambda(s)||y(s)|(\psi_k)\Big]\Bigg)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{1}{\rho^{q_1+q_2}\Gamma(q_1+q_2)}\int_a^T (T-s)^{q_1+q_2-1}(2L_1R_1+L_2\phi_0(s-a)R_1+M_1)ds \\
& +R_{1a}I^{q_2,\rho}|\lambda(s)|(T)+\frac{1}{|\Omega|}\Bigg[\left(\frac{|\Omega_4|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_3|\right) \\
& \times\Bigg(\sum_{j=1}^{n}|\alpha_j|\Bigg[\frac{1}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j)}\int_a^{\eta_j}(\eta_j-s)^{q_1+q_2+\beta_j-1} \\
& \times(2L_1R_1+L_2\phi_0(s-a)R_1+M_1)ds+R_{1a}I^{q_2+\beta_j,\rho}|\lambda(s)|(\eta_j)\Bigg] \\
& +\sum_{i=1}^{m}|\kappa_i|\Bigg[\frac{1}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i)} \\
& \times\int_a^{\sigma_i}(\sigma_i-s)^{q_1+q_2+\mu_i-1}(2L_1R_1+L_2\phi_0(s-a)R_1+M_1)ds \\
& +R_{1a}I^{q_2+\mu_i,\rho}|\lambda(s)|(\sigma_i)\Bigg]\Bigg)+\left(|\Omega_1|+\frac{|\Omega_2|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right) \\
& \times\Bigg(\sum_{l=1}^{r}|\nu_l|\Bigg[\frac{1}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l)}\int_a^{\xi_l}(\xi_l-s)^{q_1+q_2+\varphi_l-1} \\
& \times(2L_1R_1+L_2\phi_0(s-a)R_1+M_1)ds+R_{1a}I^{q_2+\varphi_l,\rho}|\lambda(s)|(\xi_l)\Bigg] \\
& +\sum_{k=1}^{p}|\omega_k|\Bigg[\frac{1}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k)} \\
& \times\int_a^{\psi_k}(\psi_k-s)^{q_1+q_2+\gamma_k-1}(2L_1R_1+L_2\phi_0(s-a)R_1+M_1)ds \\
& +R_{1a}I^{q_2+\gamma_k,\rho}|\lambda(s)|(\psi_k)\Bigg]\Bigg)\Bigg] \\
\leq\ & (2L_1R_1+M_1)\Bigg[\frac{(T-a)^{q_1+q_2}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)}+\Lambda_1\Bigg(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)} \\
& +\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\Bigg)+\Lambda_2\Bigg(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)} \\
& +\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\Bigg)\Bigg]+L_2\phi_0R_1\Bigg[\frac{(T-a)^{q_1+q_2+1}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+2)} \\
& +\Lambda_1\Bigg(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j+1}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+2)}+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i+1}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+2)}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
& +\Lambda_2\left(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l+1}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+2)}+\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k+1}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+2)}\right)\Bigg] \\
& +R_1\Bigg[{}_aI^{q_2,\rho}|\lambda(s)|(T)+\Lambda_1\Bigg((\sum_{j=1}^{n}|\alpha_j|{}_aI^{q_2+\beta_j,\rho}|\lambda(s)|(\eta_j) \\
& +\sum_{i=1}^{m}|\kappa_i|{}_aI^{q_2+\mu_i,\rho}|\lambda(s)|(\sigma_i)\Bigg))+\Lambda_2\Bigg(\sum_{l=1}^{r}|\nu_l|{}_aI^{q_2+\varphi_l,\rho}|\lambda(s)|(\xi_l) \\
& +\sum_{k=1}^{p}|\omega_k|{}_aI^{q_2+\gamma_k,\rho}|\lambda(s)|(\psi_k)\Bigg)\Bigg] \\
\leq\ & (2L_1R_1+M_1)\Lambda_3(u)+L_2\phi_0R_1\Lambda_3(u+1)+R_1\Lambda_4\leq R_1,
\end{aligned}
$$

which implies that $\|\mathcal{Q}y\|\leq R_1$. Thus, $\mathcal{Q}B_{R_1}\subset B_{R_1}$.

Step II. To present that an operator $\mathcal{Q}:\mathbb{E}\to\mathbb{E}$ is a contraction mapping.

For all y, $z\in\mathbb{E}$ and for each $t\in[a,T]$, we obtain

$$
\begin{aligned}
& |(\mathcal{Q}y)(t)-(\mathcal{Q}z)(t)| \\
\leq\ & {}_aI^{q_1+q_2,\rho}|F_y(s)-F_z(s)|(T)+{}_aI^{q_2,\rho}|\lambda(s)||y(s)-z(s)|(T) \\
& +\frac{e^{\frac{\rho-1}{\rho}(T-a)}}{|\Omega|}\Bigg[\left(\frac{|\Omega_4|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_3|\right) \\
& \times\Bigg(\sum_{j=1}^{n}|\alpha_j|\Big[{}_aI^{q_1+q_2+\beta_j,\rho}|F_y(s)-F_z(s)|(\eta_j)+{}_aI^{q_2+\beta_j,\rho}|\lambda(s)||y(s)-z(s)|(\eta_j)\Big] \\
& +\sum_{i=1}^{m}|\kappa_i|\Big[{}_aI^{q_1+q_2+\mu_i,\rho}|F_y(s)-F_z(s)|(\sigma_i)+{}_aI^{q_2+\mu_i,\rho}|\lambda(s)||y(s)-z(s)|(\sigma_i\Big]\Bigg) \\
& +\left(\frac{|\Omega_2|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_1|\right)\Bigg(\sum_{l=1}^{r}|\nu_l|\Big[{}_aI^{q_1+q_2+\varphi_l,\rho}|F_y(s)-F_z(s)|(\xi_l) \\
& +{}_aI^{q_2+\varphi_l,\rho}|\lambda(s)||y(s)-z(s)|(\xi_l)\Big]+\sum_{k=1}^{p}|\omega_k|\Big[{}_aI^{q_1+q_2+\gamma_k,\rho}|F_y(s)-F_z(s)|(\psi_k) \\
& +{}_aI^{q_2+\gamma_k,\rho}|\lambda(s)||y(s)-z(s)|(\psi_k)\Big]\Bigg)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & {}_aI^{q_1+q_2,\rho}(2L_1+L_2\phi_0(s-a))(T)\|y-z\| + {}_aI^{q_2,\rho}|\lambda(s)|(T)\|y-z\| \\
&+\frac{e^{\frac{\rho-1}{\rho}(T-a)}}{|\Omega|}\Bigg[\left(\frac{|\Omega_4|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_3|\right) \\
&\times\Bigg(\sum_{j=1}^{n}|\alpha_j|\Big[{}_aI^{q_1+q_2+\beta_j,\rho}(2L_1+L_2\phi_0(s-a))(\eta_j)\|y-z\| \\
&+{}_aI^{q_2+\beta_j,\rho}|\lambda(s)|(\eta_j)\|y-z\|\Big] \\
&+\sum_{i=1}^{m}|\kappa_i|\Big[{}_aI^{q_1+q_2+\mu_i,\rho}(2L_1+L_2\phi_0(s-a))(\sigma_i)\|y-z\| \\
&+{}_aI^{q_2+\mu_i,\rho}|\lambda(s)|(\sigma_i)\|y-z\|\Big]\Bigg)+\left(\frac{|\Omega_2|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_1|\right) \\
&\times\Bigg(\sum_{l=1}^{r}|\nu_l|\Big[{}_aI^{q_1+q_2+\varphi_l,\rho}(2L_1+L_2\phi_0(s-a))(\xi_l)\|y-z\| \\
&+{}_aI^{q_2+\varphi_l,\rho}|\lambda(s)|(\xi_l)\|y-z\|\Big] \\
&+\sum_{k=1}^{p}|\omega_k|\Big[{}_aI^{q_1+q_2+\gamma_k,\rho}(2L_1+L_2\phi_0(s-a))(\psi_k)\|y-z\| \\
&+{}_aI^{q_2+\gamma_k,\rho}|\lambda(s)|(\psi_k)\|y-z\|\Big]\Bigg)\Bigg] \\
\leq\ & \Bigg[2L_1\Bigg[\frac{(T-a)^{q_1+q_2}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)}+\Lambda_1\Bigg(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)} \\
&+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\Bigg)+\Lambda_2\Bigg(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)} \\
&+\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\Bigg)\Bigg]+L_2\phi_0\Bigg[\frac{(T-a)^{q_1+q_2+1}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+2)} \\
&+\Lambda_1\Bigg(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j+1}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+2)}+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i+1}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+2)}\Bigg) \\
&+\Lambda_2\Bigg(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l+1}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+2)}+\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k+1}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+2)}\Bigg)\Bigg] \\
&+{}_aI^{q_2,\rho}|\lambda(s)|(T)+\Lambda_1\left(\sum_{j=1}^{n}|\alpha_j|{}_aI^{q_2+\beta_j,\rho}|\lambda(s)|(\eta_j)+\sum_{i=1}^{m}|\kappa_i|{}_aI^{q_2+\mu_i,\rho}|\lambda(s)|(\sigma_i)\right) \\
&+\Lambda_2\left(\sum_{l=1}^{r}|\nu_l|{}_aI^{q_2+\varphi_l,\rho}|\lambda(s)|(\xi_l)+\sum_{k=1}^{p}|\omega_k|{}_aI^{q_2+\gamma_k,\rho}|\lambda(s)|(\psi_k)\right)\Bigg]\|y-z\|
\end{aligned}
$$

$$\leq \quad [2L_1\Lambda_3(q_1+q_2) + L_2\phi_0\Lambda_3(q_1+q_2+1) + \Lambda_4] \, \|y-z\|,$$

This implies that $\|\mathcal{Q}y-\mathcal{Q}z\| \leq [2L_1\Lambda_3(q_1+q_2)+L_2\phi_0\Lambda_3(q_1+q_2+1)+\Lambda_4] \, \|y-z\|$. As $2L_1\Lambda_3(q_1+q_2)+L_2\phi_0\Lambda_3(q_1+q_2+1)+\Lambda_4 < 1$, hence, by the Banach's fixed point theorem (Lemma 2.33), the operator $\mathcal{Q}$ is a contraction mapping. Thus, it has a unique fixed point in $\mathbb{E}$. The proof is completed. □

3.2.3 Existence result via Krasnoselskii's Fixed Point Theorem

Theorem 3.7. Suppose that (H_1), (H_2), (H_4) and (H_5) hold. Then the problem (1.2) has at least one solution on $[a,T]$ provided $\Lambda_4 < 1$, where Λ_4 is defined by (3.26).

Proof. Let $\sup_{t\in[a,T]} |g(t)| = \|g\|$, and $B_{R_2} = \{y \in \mathbb{E} : \|y\| \leq R_2\}$, where

$$R_2 \geq \frac{\Lambda_3(q_1+q_2)\|g\|}{1-\Lambda_4} \tag{3.29}$$

and $\Lambda_3(q_1+q_2)$, Λ_4 are defined by (3.25), (3.26), respectively, we determine the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on B_{R_2} by

$$\begin{aligned}(\mathcal{Q}_1y)(t) \quad = \quad & {}_aI^{q_1+q_2,\rho}F_y(s)(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\left[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right)\right.\\ & \times\left(\sum_{j=1}^{n}\alpha_{j\,a}I^{q_1+q_2+\beta_j,\rho}F_y(s)(\eta_j) - \sum_{i=1}^{m}\kappa_{i\,a}I^{q_1+q_2+\mu_i,\rho}F_y(s)(\sigma_i)\right)\\ & + \left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\left(\sum_{l=1}^{r}\nu_{l\,a}I^{q_1+q_2+\varphi_l,\rho}F_y(s)(\xi_l)\right.\\ & \left.\left. - \sum_{k=1}^{p}\omega_{k\,a}I^{q_1+q_2+\gamma_k,\rho}F_y(s)(\psi_k)\right)\right], \quad t \in [a,T],\end{aligned}$$

$$
\begin{aligned}
(\mathcal{Q}_2 y)(t) &= \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}-\Omega_3\right)\left(\sum_{i=1}^{m}\kappa_{i\,a}I^{q_2+\mu_i,\rho}\lambda(s)y(s)(\sigma_i)\right.\\
&\quad\left.-\sum_{j=1}^{n}\alpha_{j\,a}I^{q_2+\beta_j,\rho}\lambda(s)y(s)(\eta_j)\right)+\left(\Omega_1-\frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\\
&\quad\times\left(\sum_{k=1}^{p}\omega_{k\,a}I^{q_2+\gamma_k,\rho}\lambda(s)y(s)(\psi_k)-\sum_{l=1}^{r}\nu_{l\,a}I^{q_2+\varphi_l,\rho}\lambda(s)y(s)(\xi_l)\right)\Bigg]\\
&\quad-{}_aI^{q_2,\rho}\lambda(s)x(s)(t),\quad t\in[a,T].
\end{aligned}
$$

We have $\mathcal{Q}=\mathcal{Q}_1+\mathcal{Q}_2$.

For any y, $z\in B_{R_2}$, we have

$$
\begin{aligned}
&\|\mathcal{Q}_1 y+\mathcal{Q}_2 z\|\\
\leq\;& \sup_{t\in[a,T]}\Bigg\{{}_aI^{q_1+q_2,\rho}|F_y(s)|(t)+{}_aI^{q_2,\rho}|\lambda(s)||z(s)|(t)\\
&+\frac{e^{\frac{\rho-1}{\rho}(t-a)}}{|\Omega|}\Bigg[\left(\frac{|\Omega_4|(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_3|\right)\\
&\times\left(\sum_{j=1}^{n}|\alpha_j|\left[{}_aI^{q_1+q_2+\beta_j,\rho}|F_y(s)|(\eta_j)+{}_aI^{q_2+\beta_j,\rho}|\lambda(s)||z(s)|(\eta_j)\right]\right.\\
&\left.+\sum_{i=1}^{m}|\kappa_i|\left[{}_aI^{q_1+q_2+\mu_i,\rho}|F_y(s)|(\sigma_i)+{}_aI^{q_2+\mu_i,\rho}|\lambda(s)||z(s)|(\sigma_i)\right]\right)\\
&+\left(|\Omega_1|+\frac{|\Omega_2|(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\left(\sum_{l=1}^{r}|\nu_l|\left[{}_aI^{q_1+q_2+\varphi_l,\rho}|F_y(s)|(\xi_l)\right.\right.\\
&\left.+{}_aI^{q_2+\varphi_l,\rho}|\lambda(s)||z(s)|(\xi_l)\right]\\
&\left.+\sum_{k=1}^{p}|\omega_k|\left[{}_aI^{q_1+q_2+\gamma_k,\rho}|F_y(s)|(\psi_k)+{}_aI^{q_2+\gamma_k,\rho}|\lambda(s)||z(s)|(\psi_k)\right]\right)\Bigg]\Bigg\}\\
\leq\;& \|g\|\Bigg[\frac{(T-a)^{q_1+q_2}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)}\\
&+\Lambda_1\left(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)}+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i})}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\right)\\
&+\Lambda_2\left(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)}+\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\right)\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+R_2\Bigg[{}_aI^{q_2,\rho}|\lambda(s)|(T)\\
&+\Lambda_1\left(\sum_{j=1}^{n}|\alpha_j|{}_aI^{q_2+\beta_j,\rho}|\lambda(s)|(\eta_j)+\sum_{i=1}^{m}|\kappa_i|{}_aI^{q_2+\mu_i,\rho}|\lambda(s)|(\sigma_i)\right)\\
&+\Lambda_2\left(\sum_{l=1}^{r}|\nu_l|{}_aI^{q_2+\varphi_l,\rho}|\lambda(s)|(\xi_l)+\sum_{k=1}^{p}|\omega_k|{}_aI^{q_2+\gamma_k,\rho}|\lambda(s)|(\psi_k)\right)\Bigg]\\
=\;&\Lambda_3(q_1+q_2)\|g\|+\Lambda_4R_2\le R_2.
\end{aligned}
$$

which implies that $\|\mathcal{Q}_1y+\mathcal{Q}_2z\|\le R_2$. It follows that $\mathcal{Q}_1y+\mathcal{Q}_2z\in B_{R_2}$, which satisfies the assumption (i) of Lemma 2.35.

To show that the assumption (ii) of Lemma 2.35 is satisfied, let $\{y_n\}$ be a sequence such that $y_n\to y$ in $\mathbb{E}$. Then, for each $t\in[a,T]$, we take

$$
\begin{aligned}
&|(\mathcal{Q}_1y_n)(t)-(\mathcal{Q}_1y)(t)|\\
\le\;&{}_aI^{q_1+q_2,\rho}|F_{y_n}(s)-F_y(s)|(T)+\frac{1}{|\Omega|}\Bigg[\left(\frac{|\Omega_4|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_3|\right)\\
&\times\left(\sum_{j=1}^{n}|\alpha_j|{}_aI^{q_1+q_2+\beta_j,\rho}|F_{y_n}(s)-F_y(s)|(\eta_j)\right.\\
&\left.+\sum_{i=1}^{m}|\kappa_i|{}_aI^{q_1+q_2+\mu_i,\rho}|F_{y_n}(s)-F_y(s)|(\sigma_i)\right)\\
&+\left(|\Omega_1|+\frac{|\Omega_2|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\left(\sum_{l=1}^{r}|\nu_l|{}_aI^{q_1+q_2+\varphi_l,\rho}|F_{y_n}(s)-F_y(s)|(\xi_l)\right.\\
&\left.+\sum_{k=1}^{p}|\omega_k|{}_aI^{q_1+q_2+\gamma_k,\rho}|F_{y_n}(s)-F_y(s)|(\psi_k)\right)\Bigg]\\
\le\;&\Bigg\{\frac{(T-a)^{q_1+q_2}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)}\\
&+\Lambda_1\left(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)}+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\right)\\
&+\Lambda_2\left(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)}+\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\right)\Bigg\}\\
&\times\|F_{y_n}-F_y\|
\end{aligned}
$$

$$= \Lambda_3(q_1+q_2)\|F_{y_n} - F_y\|.$$

Since the functions f_2 and λ are continuous, by the Lebesgue dominated convergent theorem, we have

$$|(\mathcal{Q}_1 y_n)(t) - (\mathcal{Q}_1 y)(t)| \to 0, \quad \text{as} \quad n \to \infty.$$

Hence,

$$\|\mathcal{Q}_1 y_n - \mathcal{Q}_1 y\| \to 0, \quad \text{as} \quad n \to \infty.$$

Therefore, the operator $\mathcal{Q}_1$ is continuous. Also, the set $\mathcal{Q}_1 B_{R_2}$ is uniformly bounded as

$$\|\mathcal{Q}_1 y\| \leq \Lambda_3(q_1+q_2)\|g\|.$$

Next, we prove the compactness of the operator $\mathcal{Q}_1$.

Setting $\sup_{(t,x,y,z)\in[a,T]\times B^3_{R_2}} |f_2(t,x,y,z)| = f^* < \infty$, for each τ_1, $\tau_2 \in [a,T]$ with $a \leq \tau_1 \leq \tau_2 \leq T$, we have

$$\begin{aligned}
& |(\mathcal{Q}_1 y)(\tau_2) - (\mathcal{Q}_1 y)(\tau_1)| \\
\leq \; & \left| {}_aI^{q_1+q_2,\rho}F_y(s)(\tau_2) - {}_aI^{q_1+q_2,\rho}F_y(s)(\tau_1) \right| + \frac{1}{|\Omega|} \left| e^{\frac{\rho-1}{\rho}(\tau_2-a)} - e^{\frac{\rho-1}{\rho}(\tau_1-a)} \right| \\
& \times \Bigg[|\Omega_4| \left(\frac{(\tau_2-a)^{q_2} - (\tau_1-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} \right) \left(\sum_{j=1}^{n} |\alpha_j| {}_aI^{q_1+q_2+\beta_j,\rho}F_y(s)(\eta_j) \right. \\
& \left. + \sum_{i=1}^{m} |\kappa_i| {}_aI^{q_1+q_2+\mu_i,\rho}F_y(s)(\sigma_i) \right) + |\Omega_2| \left(\frac{(\tau_2-a)^{q_2} - (\tau_1-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} \right) \\
& \times \left(\sum_{l=1}^{r} |\nu_l| {}_aI^{q_1+q_2+\varphi_l,\rho}F_y(s)(\xi_l) + \sum_{k=1}^{p} |\omega_k| {}_aI^{q_1+q_2+\gamma_k,\rho}F_y(s)(\psi_k) \right) \Bigg]
\end{aligned}$$

$$
\begin{aligned}
\leq\ & f^*\Bigg\{\frac{1}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)}\Big(\left|(\tau_2-a)^{q_1+q_2}-(\tau_1-a)^{q_1+q_2}-(\tau_2-\tau_1)^{q_1+q_2}\right| \\
& +(\tau_2-\tau_1)^{q_1+q_2}\Big) \\
& +\frac{1}{|\Omega|}\left|e^{\frac{\rho-1}{\rho}(\tau_2-a)}-e^{\frac{\rho-1}{\rho}(\tau_1-a)}\right|\Bigg[|\Omega_4|\left(\frac{(\tau_2-a)^{q_2}-(\tau_1-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right) \\
& \times\left(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)}\right. \\
& \left.+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\right)+|\Omega_2|\left(\frac{(\tau_2-a)^{q_2}-(\tau_1-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right) \\
& \times\left(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)}+\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\right)\Bigg]\Bigg\},
\end{aligned}
$$

which is independent of y and $|(\mathcal{Q}_1y)(\tau_2)-(\mathcal{Q}_1y)(\tau_1)|\to 0$ as $\tau_2\to\tau_1$. Thus, the set $\mathcal{Q}_1B_{R_2}$ is equicontinuous, the operator $\mathcal{Q}_1$ maps bounded subsets into relatively compact subsets, it follows that the set $\mathcal{Q}_1B_{R_2}$ is relatively compact. Then, by the Arzelá-Ascoli theorem, the operator $\mathcal{Q}_1$ is compact on B_{R_2}. It is easy to see that, using $\Lambda_4<1$, the operator $\mathcal{Q}_2$ is a contraction mapping and also the assumption (iii) of Lemma 2.35 holds. Thus, all assumptions of Lemma 2.35 are satisfied. Hence, the conclusion of Theorem 3.7 implies that the problem (1.2) has at least one solution on $[a,T]$. This completes the proof. □

3.2.4 Existence result via Schaefer's Fixed Point Theorem

Theorem 3.8. Suppose that (H_1), (H_2), (H_4) and (H_6) satisfied. Then, the problem (1.2) has at least one solution on $[a,T]$.

Proof. We will show that the operator $\mathcal{Q}$ has at least one fixed point in $\mathbb{E}$. The proof is divided into a sequence of four steps.

Step I The operator $\mathcal{Q}$ is continuous.

Let y_n be a sequence such that $y_n\to y$ in $\mathbb{E}$. Then for each $t\in[a,T]$, we obtain

$$
\begin{aligned}
& |(\mathcal{Q}y_n)(t) - (\mathcal{Q}y)(t)| \\
\leq\ & {}_aI^{q_1+q_2,\rho}|F_{y_n}(s) - F_y(s)|(t) + {}_aI^{q_2,\rho}|\lambda(s)||y_n(s) - y(s)|(t) \\
& + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{|\Omega|}\Bigg[\left(\frac{|\Omega_4|(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} + |\Omega_3|\right)\left(\sum_{j=1}^{n}|\alpha_j|\Big[{}_aI^{q_1+q_2+\beta_j,\rho}|F_{y_n}(s) - F_y(s)|(\eta_j)\right. \\
& + {}_aI^{q_2+\beta_j,\rho}|\lambda(s)||y_n(s) - y(s)|(\eta_j)\Big] + \sum_{i=1}^{m}|\kappa_i|\Big[{}_aI^{q_1+q_2+\mu_i,\rho}|F_{y_n}(s) - F_y(s)|(\sigma_i) \\
& \left. + {}_aI^{q_2+\mu_i,\rho}|\lambda(s)||y_n(s) - y(s)|(\sigma_i)\Big]\right) + \left(|\Omega_1| + \frac{|\Omega_2|(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right) \\
& \times\left(\sum_{l=1}^{r}|\nu_l|\Big[{}_aI^{q_1+q_2+\varphi_l,\rho}|F_{y_n}(s) - F_y(s)|(\xi_l) + {}_aI^{q_2+\varphi_l,\rho}|\lambda(s)||y_n(s) - y(s)|(\xi_l)\Big]\right. \\
& \left. + \sum_{k=1}^{p}|\omega_k|\Big[{}_aI^{q_1+q_2+\gamma_k,\rho}|F_{y_n}(s) - F_y(s)|(\psi_k) + {}_aI^{q_2+\gamma_k,\rho}|\lambda(s)||y_n(s) - y(s)|(\psi_k)\Big]\right)\Bigg] \\
\leq\ & \Bigg[\frac{(T-a)^{q_1+q_2}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)} \\
& + \Lambda_1\left(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)} + \sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\right) \\
& + \Lambda_2\left(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)} + \sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\right)\Bigg] \\
& \times\|F_{y_n} - F_y\| \\
& + \Bigg[{}_aI^{q_2,\rho}|\lambda(s)|(T) + \Lambda_1\left(\sum_{j=1}^{n}|\alpha_j|{}_aI^{q_2+\beta_j,\rho}|\lambda(s)|(\eta_j) + \sum_{i=1}^{m}|\kappa_i|{}_aI^{q_2+\mu_i,\rho}|\lambda(s)|(\sigma_i)\right) \\
& + \Lambda_2\left(\sum_{l=1}^{r}|\nu_l|{}_aI^{q_2+\varphi_l,\rho}|\lambda(s)|(\xi_l) + \sum_{k=1}^{p}|\omega_k|{}_aI^{q_2+\gamma_k,\rho}|\lambda(s)|(\psi_k)\right)\Bigg]\|y_n - y\|_{\mathbb{E}} \\
=\ & \Lambda_3(q_1+q_2)\|F_{y_n} - F_y\| + \Lambda_4\|y_n - y\|.
\end{aligned}
$$

Since f and λ are continuous, this implies that the operator $\mathcal{Q}$ is also continuous. Hence, we obtain

$$\|F_{y_n} - F_y\| \to 0 \quad \text{and} \quad \|y_n - y\| \to 0, \quad \text{as} \quad n \to \infty.$$

Step II The operator $\mathcal{Q}$ maps bounded set into bounded set in $\mathbb{E}$.

For $R_3 > 0$, there is a constant $M_3 > 0$ such that, for each $y \in B_{R_3} = \{y \in \mathbb{E} : \|y\| \leq R_3\}$, we have $\|\mathcal{Q}y\| \leq M_3$. Then, for any $t \in [a, T]$ and $y \in B_{R_3}$, we have

$$
\begin{aligned}
& |(\mathcal{Q}y)(t)| \\
\leq\ & {}_aI^{q_1+q_2,\rho}|F_y(s)|(t) + {}_aI^{q_2,\rho}|\lambda(s)||y(s)|(t) \\
& + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{|\Omega|}\left[\left(\frac{|\Omega_4|(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} + |\Omega_3|\right)\left(\sum_{j=1}^{n}|\alpha_j|\Big[{}_aI^{q_1+q_2+\beta_j,\rho}|F_y(s)|(\eta_j)\right.\right. \\
& + {}_aI^{q_2+\beta_j,\rho}|\lambda(s)||y(s)|(\eta_j)\Big] \\
& \left. + \sum_{i=1}^{m}|\kappa_i|\Big[{}_aI^{q_1+q_2+\mu_i,\rho}|F_y(s)|(\sigma_i) + {}_aI^{q_2+\mu_i,\rho}|\lambda(s)||y(s)|(\sigma_i)\Big]\right) \\
& + \left(|\Omega_1| + \frac{|\Omega_2|(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right) \\
& \times\left(\sum_{l=1}^{r}|\nu_l|\left[{}_aI^{q_1+q_2+\varphi_l,\rho}|F_y(s)|(\xi_l) + {}_aI^{q_2+\varphi_l,\rho}|\lambda(s)||y(s)|(\xi_l)\right]\right. \\
& \left.\left. + \sum_{k=1}^{p}|\omega_k|\left[{}_aI^{q_1+q_2+\gamma_k,\rho}|F_y(s)|(\psi_k) + {}_aI^{q_2+\gamma_k,\rho}|\lambda(s)||y(s)|(\psi_k)\right]\right)\right].
\end{aligned}
$$

It follows from hypotheses (H_4) and (H_6) that

$$
\begin{aligned}
{}_aI^{u,\rho}|F_y(s)|(z) &\leq {}_aI^{u,\rho}\left(h_1(s) + h_2(s)|y(s)| + h_3(s)|y(\epsilon s) + h_4(s)|(\mathcal{K}y)(s)|\right)(z) \\
&\leq \frac{1}{\rho^u\Gamma(u)}\int_a^z (z-s)^{u-1}\left(h_1^* + h_2^*R_3 + h_3^*R_3 + h_4^*R_3\phi_0(s-a)\right)ds \\
&\leq (h_1^* + h_2^*R_3 + h_3^*R_3)\frac{(z-a)^u}{\rho^u\Gamma(u+1)} + h_4^*R_3\phi_0\frac{(z-a)^{u+1}}{\rho^u\Gamma(u+2)}, \qquad (3.30)
\end{aligned}
$$

where $u \in \{q_1+q_2, q_1+q_2+\mu_i, q_1+q_2+\beta_j, q_1+q_2+\varphi_l, q_1+q_2+\gamma_k\}$ and $z \in \{t, T, \sigma_i, \eta_j, \xi_l, \psi_k\}$.

This implies that

$$
\begin{aligned}
& |(\mathcal{Q}y)(t)| \\
\leq\ & \Bigg[\frac{(T-a)^{q_1+q_2}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)} \\
& +\Lambda_1\left(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)}+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\right) \\
& +\Lambda_2\left(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)}+\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\right)\Bigg] \\
& \times(h_1^*+h_2^*R_3+h_3^*R_3)+\Bigg[\frac{(T-a)^{q_1+q_2+1}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+2)} \\
& +\Lambda_1\left(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j+1}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+2)}+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i+1}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+2)}\right) \\
& +\Lambda_2\Bigg(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l+1}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+2)} \\
& +\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k+1}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+2)}\Bigg)\Bigg]h_4^*R_3\phi_0 \\
& +\Bigg[{}_aI^{q_2,\rho}|\lambda(s)|(T)+\Lambda_1\left(\sum_{j=1}^{n}|\alpha_j|{}_aI^{q_2+\beta_j,\rho}|\lambda(s)|(\eta_j)+\sum_{i=1}^{m}|\kappa_i|{}_aI^{q_2+\mu_i,\rho}|\lambda(s)|(\sigma_i)\right) \\
& +\Lambda_2\left(\sum_{l=1}^{r}|\nu_l|{}_aI^{q_2+\varphi_l,\rho}|\lambda(s)|(\xi_l)+\sum_{k=1}^{p}|\omega_k|{}_aI^{q_2+\gamma_k,\rho}|\lambda(s)|(\psi_k)\right)\Bigg]R_3 \\
\leq\ & \Lambda_3(q_1+q_2)(h_1^*+h_2^*R_3+h_3^*R_3)+\Lambda_3(q_1+q_2+1)h_4^*R_3\phi_0+\Lambda_4R_3:=M_3,
\end{aligned}
$$

we estimate

$$\|\mathcal{Q}y\|\leq\Lambda_3(q_1+q_2)(h_1^*+h_2^*R_3+h_3^*R_3)+\Lambda_3(q_1+q_2+1)h_4^*R_3\phi_0+\Lambda_4R_3:=M_3.$$

where Λ_1, Λ_2, $\Lambda_3(u)$, $u\in\{q_1+q_2, q_1+q_2+1\}$ and Λ_4 are given by (3.23), (3.24), (3.25) and (3.26), respectively.

Step III The operator $\mathcal{Q}$ maps bounded set into equicontinuous set of $\mathbb{E}$.

For $a\leq\tau_1<\tau_2\leq T$ and $y\in B_{R_3}$ where B_{R_3} as defined in Step II. By using the property f_2 is bounded on the compact set $[a,T]\times B_{R_3}$, we obtain

$$
\begin{aligned}
& |(\mathcal{Q}y)(\tau_2)-(\mathcal{Q}y)(\tau_1)| \\
\leq\ & \left|{}_aI^{q_1+q_2,\rho}F_y(s)(\tau_2)-{}_aI^{q_1+q_2,\rho}F_y(s)(\tau_1)\right| \\
& +|{}_aI^{q_2,\rho}\lambda(s)y(s)(\tau_2)-{}_aI^{q_2,\rho}\lambda(s)y(s)(\tau_1)| \\
& +\frac{1}{|\Omega|}\left|e^{\frac{\rho-1}{\rho}(\tau_2-a)}-e^{\frac{\rho-1}{\rho}(\tau_1-a)}\right|\Bigg[|\Omega_4|\left(\frac{(\tau_2-a)^{q_2}-(\tau_1-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right) \\
& \times\Bigg(\sum_{j=1}^{n}|\alpha_j|\left[{}_aI^{q_1+q_2+\beta_j,\rho}|F_y(s)|(\eta_j)+{}_aI^{q_2+\beta_j,\rho}|\lambda(s)||y(s)|(\eta_j)\right] \\
& +\sum_{i=1}^{m}|\kappa_i|\left[{}_aI^{q_1+q_2+\mu_i,\rho}|F_y(s)|(\sigma_i)+{}_aI^{q_2+\mu_i,\rho}|\lambda(s)||y(s)|(\sigma_i)\right]\Bigg) \\
& +|\Omega_2|\left(\frac{(\tau_2-a)^{q_2}-(\tau_1-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right) \\
& \times\Bigg(\sum_{l=1}^{r}|\nu_l|\left[{}_aI^{q_1+q_2+\varphi_l,\rho}F_y(s)(\xi_l)+{}_aI^{q_2+\varphi_l,\rho}|\lambda(s)||y(s)|(\xi_l)\right] \\
& +\sum_{k=1}^{p}|\omega_k|\left[{}_aI^{q_1+q_2+\gamma_k,\rho}F_y(s)(\psi_k)+{}_aI^{q_2+\gamma_k,\rho}|\lambda(s)||y(s)|(\psi_k)\right]\Bigg)\Bigg] \\
\leq\ & \frac{1}{\rho^{q_1+q_2}\Gamma(q_1+q_2)}\int_a^{\tau_1}\left|e^{\frac{\rho-1}{\rho}(\tau_2-s)}(\tau_2-s)^{q_1+q_2-1}-e^{\frac{\rho-1}{\rho}(\tau_1-s)}(\tau_1-s)^{q_1+q_2-1}\right| \\
& \times(h_1^*+h_2^*R_3+h_3^*R_3+h_4^*R_3\phi_0(s-a))ds \\
& +\frac{1}{\rho^{q_1+q_2}\Gamma(q_1+q_2)}\int_{\tau_1}^{\tau_2}e^{\frac{\rho-1}{\rho}(\tau_2-s)}(\tau_2-s)^{q_1+q_2-1} \\
& \times(h_1^*+h_2^*R_3+h_3^*R_3+h_4^*R_3\phi_0(s-a))ds \\
& +\frac{R_3}{\rho^{q_2}\Gamma(q_2)}\int_{\tau_1}^{\tau_2}e^{\frac{\rho-1}{\rho}(\tau_2-s)}(\tau_2-s)^{q_1+q_2-1}|\lambda(s)|ds \\
& +\frac{R_3}{\rho^{q_2}\Gamma(q_2)}\int_a^{\tau_1}\left|e^{\frac{\rho-1}{\rho}(\tau_2-s)}(\tau_2-s)^{q_1+q_2-1}-e^{\frac{\rho-1}{\rho}(\tau_1-s)}(\tau_1-s)^{q_1+q_2-1}\right||\lambda(s)|ds \\
& +\frac{h_1^*+h_2^*R_3+h_3^*R_3+h_4^*R_3\phi_0}{|\Omega|}\left|e^{\frac{\rho-1}{\rho}(\tau_2-a)}-e^{\frac{\rho-1}{\rho}(\tau_1-a)}\right| \\
& \times\Bigg[|\Omega_4|\left(\frac{(\tau_2-a)^{q_2}-(\tau_1-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)
\end{aligned}
$$

$$\times\left(\sum_{j=1}^{n}|\alpha_j|\left(\frac{(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)}+\frac{(\eta_j-a)^{q_1+q_2+\beta_j+1}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+2)}\right)\right.$$
$$\left.+\sum_{i=1}^{m}|\kappa_i|\left(\frac{(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}+\frac{(\sigma_i-a)^{q_1+q_2+\mu_i+1}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+2)}\right)\right)$$
$$+\left(\sum_{l=1}^{r}|\nu_l|\left(\frac{(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)}+\frac{(\xi_l-a)^{q_1+q_2+\varphi_l+1}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+2)}\right)\right.$$
$$\left.+\sum_{k=1}^{p}|\omega_k|\left(\frac{(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}+\frac{(\psi_k-a)^{q_1+q_2+\gamma_k+1}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+2)}\right)\right)$$
$$\left.\times|\Omega_2|\left(\frac{(t_2-a)^{q_2}-(t_1-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\right]+R_3\left[\sum_{j=1}^{n}|\alpha_j|{}_aI^{q_2+\beta_j,\rho}|\lambda(s)|(\eta_j)\right.$$
$$+\sum_{i=1}^{m}|\kappa_i|{}_aI^{q_2+\mu_i,\rho}|\lambda(s)|(\sigma_i)+\sum_{l=1}^{r}|\nu_l|{}_aI^{q_2+\varphi_l,\rho}|\lambda(s)|(\xi_l)$$
$$\left.+\sum_{k=1}^{p}|\omega_k|{}_aI^{q_2+\gamma_k,\rho}|\lambda(s)|(\psi_k)\right].$$

The R.H.S of the above inequality tends to zero as $\tau_2\to\tau_1$, implying that $\|(\mathcal{Q}y)(\tau_2)-(\mathcal{Q}y)(\tau_1)\|\to 0$ as $\tau_2\to\tau_1$. Hence, by Steps I to III, and with the Arzelá-Ascoli theorem, we conclude that the operator $\mathcal{Q}$ is completely continuous.

Step IV The set $\mathbb{D}=\{y\in\mathbb{E}: y=\varrho\mathcal{Q}y,\ 0\le\varrho\le 1\}$ is bounded (by a priori bounds).

Let $y\in\mathbb{D}$, then $y=\varrho\mathcal{Q}y$ for some $0<\varrho<1$. From (H_4)-(H_5), for all $t\in[a,T]$, one can get the estimates

$$|(\mathcal{Q}y)(t)|=|\varrho(\mathcal{Q}y)(t)|$$
$$\le\ {}_aI^{q_1+q_2,\rho}|F_y(s)|(T)+{}_aI^{q_2,\rho}|\lambda(s)||y(s)|(T)+\frac{e^{\frac{\rho-1}{\rho}(T-a)}}{|\Omega|}\left[\left(\frac{|\Omega_4|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}+|\Omega_3|\right)\right.$$
$$\times\left(\sum_{j=1}^{n}|\alpha_j|\left[{}_aI^{q_1+q_2+\beta_j,\rho}|F_y(s)|(\eta_j)+{}_aI^{q_2+\beta_j,\rho}|\lambda(s)||y(s)|(\eta_j)\right]\right.$$
$$\left.+\sum_{i=1}^{m}|\kappa_i|\left[{}_aI^{q_1+q_2+\mu_i,\rho}|F_y(s)|(\sigma_i)+{}_aI^{q_2+\mu_i,\rho}|\lambda(s)||y(s)|(\sigma_i)\right]\right)$$
$$+\left(|\Omega_1|+\frac{|\Omega_2|(T-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)$$

$$
\begin{aligned}
& \times\left(\sum_{l=1}^{r}|\nu_l|\left[{}_aI^{q_1+q_2+\varphi_l,\rho}|F_y(s)|(\xi_l)+{}_aI^{q_2+\varphi_l,\rho}|\lambda(s)||y(s)|(\xi_l)\right]\right. \\
& \left.\left.+\sum_{k=1}^{p}|\omega_k|\left[{}_aI^{q_1+q_2+\gamma_k,\rho}|F_y(s)|(\psi_k)+{}_aI^{q_2+\gamma_k,\rho}|\lambda(s)||y(s)|(\psi_k)\right]\right)\right] \\
\leq\ & \left[\frac{(T-a)^{q_1+q_2}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)}\right. \\
& +\Lambda_1\left(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)}+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\right) \\
& \left.+\Lambda_2\left(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)}+\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\right)\right] \\
& \times(h_1^*+h_2^*R_3+h_3^*R_3)+\left[\frac{(T-a)^{q_1+q_2+1}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+2)}\right. \\
& +\Lambda_1\left(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j+1}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+2)}+\sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i+1}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+2)}\right) \\
& +\Lambda_2\left(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l+1}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+2)}\right. \\
& \left.\left.+\sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k+1}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+2)}\right)\right]h_4^*R_3\phi_0+\left[{}_aI^{q_2,\rho}|\lambda(s)|(T)\right. \\
& +\Lambda_1\left(\sum_{j=1}^{n}|\alpha_j|{}_aI^{q_2+\beta_j,\rho}|\lambda(s)|(\eta_j)+\sum_{i=1}^{m}|\kappa_i|{}_aI^{q_2+\mu_i,\rho}|\lambda(s)|(\sigma_i)\right) \\
& \left.+\Lambda_2\left(\sum_{l=1}^{r}|\nu_l|{}_aI^{q_2+\varphi_l,\rho}|\lambda(s)|(\xi_l)+\sum_{k=1}^{p}|\omega_k|{}_aI^{q_2+\gamma_k,\rho}|\lambda(s)|(\psi_k)\right)\right]R_3 \\
=\ & \Lambda_3(q_1+q_2)(h_1^*+h_2^*R_3+h_3^*R_3)+\Lambda_3(q_1+q_2+1)h_4^*R_3\phi_0+\Lambda_4R_3.
\end{aligned}
$$

Thus, $\|\mathcal{Q}y\|\leq\Lambda_3(q_1+q_2)(h_1^*+h_2^*R_3+h_3^*R_3)+\Lambda_3(q_1+q_2+1)h_4^*R_3\phi_0+\Lambda_4R_3:=N<\infty$. This implies that the set $\mathbb{D}$ is bounded.

By all hypotheses of Theorem 3.8, we conclude that there exists a positive constant N such that $\|y\|\leq N<\infty$. By applying the Schaefer's Fixed Point Theorem (Lemma 2.34), the operator $\mathcal{Q}$ has at least one fixed point. This completes the proof. □

CHAPTER 4

STABILITY RESULTS

In this chapter, we will prove the stability results for proportional fractional pantograph differential equation of problem (1.1) and proportional fractional Langevin differential equation of problem (1.2).

4.1 Stability Results for Proportional Fractional pantograph Differential Equation.

In this section, we prove the Ulam stability for the PCF pantograph differential equation with mixed nonlocal boundary conditions in problem (1.1), that is $\mathcal{UH}$ stable, $\mathcal{GUH}$ stable, $\mathcal{UHR}$ stable and $\mathcal{GUHR}$ stable.

Definition 4.1. The problem (1.1) is called $\mathcal{UH}$ stable, if there is a constant $\Phi > 0$, such that, for every $\varrho > 0$ and for each solution $z \in \mathbb{E}^1 = C^1([a,T],\mathbb{R})$ of the inequality

$$\left| {}_a^C D^{\alpha,\rho} z(t) - f_1(t, z(t), z(\lambda t), ({}_a^C D^{\alpha,\rho} z)(\lambda t)) \right| \leq \varrho, \quad t \in [a,T], \tag{4.1}$$

there exists a solution $u \in \mathbb{E}^1$ of the problem (1.1) such that

$$|z(t) - u(t)| \leq \Phi\varrho, \quad t \in [a,T]. \tag{4.2}$$

Definition 4.2. The problem (1.1) is called $\mathcal{GUH}$ stable, if there is a function $\Phi_{f_1} \in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\Phi_{f_1}(0) = 0$ such that for any $z \in \mathbb{E}^1$ a solotion of inequality (4.1) there is a solution $u \in \mathbb{E}^1$ of the problem (1.1) such that

$$|z(t) - u(t)| \leq \Phi_{f_1}(\varrho), \quad t \in [a,T]. \tag{4.3}$$

Definition 4.3. The problem (1.1) is called $\mathcal{UHR}$ stable with respect to $\Phi_{f_1} \in C([a,T],\mathbb{R}^+)$ if there is a real number $C_{f_1,\Phi} > 0$ such that for each $\varrho > 0$ and for each $z \in \mathbb{E}^1$ a solution of the inequality

$$\left| {}_a^C D^{\alpha,\rho} z(t) - f_1(t, z(t), z(\lambda t), ({}_a^C D^{\alpha,\rho} z)(\lambda t)) \right| \leq \varrho\Phi_{f_1}(t), \quad t \in [a,T], \tag{4.4}$$

there is a solution $u \in \mathbb{E}^1$ of the problem (1.1) such that

$$|z(t) - u(t)| \leq C_{f_1,\Phi}\varrho\Phi_{f_1}(t), \quad t \in [a,T]. \tag{4.5}$$

Definition 4.4. The problem (1.1) is called $\mathcal{GUHR}$ stable with respect to $\Phi_{f_1} \in C([a,T],\mathbb{R}^+)$ if there exists a real number $C_{f_1,\Phi} > 0$ such that for each solution $z \in \mathbb{E}^1$ of the inequality

$$\left|{}_a^C D^{\alpha,\rho} z(t) - f_1(t, z(t), z(\lambda t), ({}_a^C D^{\alpha,\rho} z)(\lambda t))\right| \leq \Phi_{f_1}(t), \quad t \in [a,T], \tag{4.6}$$

there is a solution $u \in \mathbb{E}^1$ of the problem (1.1) such that

$$|z(t) - u(t)| \leq C_{f_1,\Phi}\Phi_{f_1}(t), \quad t \in [a,T]. \tag{4.7}$$

Remark 4.5. A function $z \in \mathbb{E}^1$ is a solution of the inequality (4.1) iff there is a function $\Psi \in C([a,T],\mathbb{R})$, such that

(i) $|\Psi(t)| \leq \varrho, \forall t \in [a,T]$.

(ii) ${}_a^C D^{\alpha,\rho} z(t) = f_1(t, z(t), z(\lambda t), ({}_a^C D^{\alpha,\rho} z)(\lambda t)) + \Psi(t), \quad t \in [a,T]$.

By Remark 4.5, the solution of the equation

$${}_a^C D^{\alpha,\rho} z(t) = f_1(t, z(t), z(\lambda t), ({}_a^C D^{\alpha,\rho} z)(\lambda t)) + \Psi(t), \quad t \in [a,T],$$

can be given as

$$\begin{aligned} z(t) &= {}_aI^{\alpha,\rho}F_z(s)(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg(A - \sum_{i=1}^{m}\gamma_{ia}I^{\alpha,\rho}F_z(s)(\eta_i) \\ &\quad - \sum_{j=1}^{n}\kappa_{ja}I^{\alpha-\beta_j,\rho}F_z(s)(\xi_j) - \sum_{r=1}^{k}\sigma_{ra}I^{\alpha+\delta_r,\rho}F_z(s)(\theta_r)\Bigg) \\ &\quad + {}_aI^{\alpha,\rho}\Psi(s)(t) - \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg(\sum_{i=1}^{m}\gamma_{ia}I^{\alpha,\rho}\Psi(s)(\eta_i) \\ &\quad + \sum_{j=1}^{n}\kappa_{ja}I^{\alpha-\beta_j,\rho}\Psi(s)(\xi_j) + \sum_{r=1}^{k}\sigma_{ra}I^{\alpha+\delta_r,\rho}\Psi(s)(\theta_r)\Bigg). \end{aligned} \tag{4.8}$$

First of all, we propose a major lemma that is used in the proofs of $\mathcal{UH}$ stability and $\mathcal{GUH}$ stability.

Lemma 4.6. If $z \in \mathbb{E}^1$ satisfies the inequality (4.1), then the function z is a solution of the inequality

$$|z(t) - (\mathcal{K}z)(t)| \leq \Lambda\varrho, \quad 0 < \varrho \leq 1, \tag{4.9}$$

where Λ is given by (3.7).

Proof. From Remark 4.5, and (4.8), we get

$$\begin{aligned}
& |z(t) - (\mathcal{K}z)(t)| \\
= & \Bigg| {}_aI^{\alpha,\rho}\Psi(s)(t) - \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg(\sum_{i=1}^{m}\gamma_{i\,a}I^{\alpha,\rho}\Psi(s)(\eta_i) \\
& + \sum_{j=1}^{n}\kappa_{j\,a}I^{\alpha-\beta_j,\rho}\Psi(s)(\xi_j) + \sum_{r=1}^{k}\sigma_{r\,a}I^{\alpha+\delta_r,\rho}\Psi(s)(\theta_r)\Bigg)\Bigg| \\
\leq & {}_aI^{\alpha,\rho}|\Psi(s)|(T) + \frac{1}{|\Omega|}\Bigg(\sum_{i=1}^{m}|\gamma_i|_aI^{\alpha,\rho}|\Psi(s)|(\eta_i) \\
& + \sum_{j=1}^{n}|\kappa_j|_aI^{\alpha-\beta_j,\rho}|\Psi(s)|(\xi_j) + \sum_{r=1}^{k}|\sigma_r|_aI^{\alpha+\delta_r,\rho}|\Psi(s)|(\theta_r)\Bigg) \\
\leq & \varrho\Bigg[\frac{(T-a)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\Bigg(\sum_{i=1}^{m}\frac{|\gamma_i|(\eta_i-a)^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)} + \sum_{j=1}^{n}\frac{|\kappa_j|(\xi_j-a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)} \\
& + \sum_{r=1}^{k}\frac{|\sigma_r|(\theta_r-a)^{\alpha+\delta_r}}{\rho^{\alpha+\delta_r}\Gamma(\alpha+\delta_r+1)}\Bigg)\Bigg] \\
= & \Lambda\varrho,
\end{aligned}$$

□

Now, we show the $\mathcal{UH}$ stability and $\mathcal{GUH}$ stability results.

Theorem 4.7. Suppose that $f_1 : [a,T] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function. If (A_2) is satisfied with

$$\frac{2\mathbb{L}_1\Lambda}{1-\mathbb{L}_2} < 1.$$

then the problem (1.1) is $\mathcal{UH}$ stable as well as $\mathcal{GUH}$ stable on $[a,T]$.

Proof. Let $z \in \mathbb{E}^1$ be a solution of the inequality (4.1) and let u be the unique solution of the problem (1.1), By applying the triangle inequality, $|a-b| \leq |a|+|b|$, and Lemma 4.6, we have

$$\begin{aligned}
|z(t)-u(t)| &= \Bigg| z(t) - {}_aI^{\alpha,\rho}F_u(s)(t) - \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg(A - \sum_{i=1}^{m}\gamma_{ia}I^{\alpha,\rho}F_u(s)(\eta_i) \\
&\quad - \sum_{j=1}^{n}\kappa_{ja}I^{\alpha-\beta_j,\rho}F_u(s)(\xi_j) - \sum_{r=1}^{k}\sigma_{ra}I^{\alpha+\delta_r,\rho}F_u(s)(\theta_r)\Bigg)\Bigg| \\
&= |z(t)-(\mathcal{K}z)(t)+(\mathcal{K}z)(t)-(\mathcal{K}u)(t)| \\
&\leq |z(t)-(\mathcal{K}z)(t)|+|(\mathcal{K}z)(t)-(\mathcal{K}u)(t)| \\
&\leq \Lambda\varrho + \frac{2\mathbb{L}_1\Lambda_1}{1-\mathbb{L}_2}|z(t)-u(t)|,
\end{aligned}$$

where Λ is defined by (3.7). This yields that

$$|z(t)-u(t)| \leq \frac{\Lambda\varrho}{1-\frac{2\mathbb{L}_1\Lambda}{1-\mathbb{L}_2}}.$$

By setting

$$\Phi = \frac{\Lambda}{1-\frac{2\mathbb{L}_1\Lambda}{1-\mathbb{L}_2}}, \tag{4.10}$$

we obtain

$$|z(t)-u(t)| \leq \Phi\varrho.$$

Thus, the problem (1.1) is $\mathcal{UH}$ stable. Furthermore, if we set $\Phi_{f_1}(\varrho) = \Phi\varrho$ such that $\Phi_{f_1}(0)=0$, so that the problem (1.1) is $\mathcal{GUH}$ stable. This completes the proof. □

Remark 4.8. A function $z \in \mathbb{E}^1$ is a solution of the inequality (4.4) iff there is a function $\Theta \in C([a,T],\mathbb{R})$, such that

(i) $|\Theta(t)| \leq \varrho\Psi_{\Theta}(t), \forall t \in [a,T]$.

(ii) ${}^C_aD^{\alpha,\rho}z(t) = f_1(t,z(t),z(\lambda t),{}^C_aD^{\alpha,\rho}z(\lambda t)) + \Theta(t), \quad t \in [a,T]$.

By Remark 4.8, the solution of the equation

$${}^C_aD^{\alpha,\rho}z(t) = f_1(t,z(t),z(\lambda t),{}^C_aD^{\alpha,\rho}z(\lambda t)) + \Theta(t), \quad t \in [a,T],$$

can be given as

$$
\begin{aligned}
z(t) &= {}_aI^{\alpha,\rho}F_z(s)(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg(A - \sum_{i=1}^{m}\gamma_{ia}I^{\alpha,\rho}F_z(s)(\eta_i) \\
&\quad - \sum_{j=1}^{n}\kappa_{ja}I^{\alpha-\beta_j,\rho}F_z(s)(\xi_j) - \sum_{r=1}^{k}\sigma_{ra}I^{\alpha+\delta_r,\rho}F_z(s)(\theta_r)\Bigg) \\
&\quad + {}_aI^{\alpha,\rho}\Theta(s)(t) - \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg(\sum_{i=1}^{m}\gamma_{ia}I^{\alpha,\rho}\Theta(s)(\eta_i) \\
&\quad + \sum_{j=1}^{n}\kappa_{ja}I^{\alpha-\beta_j,\rho}\Psi(s)(\xi_j) + \sum_{r=1}^{k}\sigma_{ra}I^{\alpha+\delta_r,\rho}\Theta(s)(\theta_r)\Bigg). \qquad (4.11)
\end{aligned}
$$

Lemma 4.9. Let $z \in \mathbb{E}^1$ be a solution of inequality (4.4). Then the function z satisfies the inequality

$$|z(t) - (\mathcal{Q}z)(t)| \le \Lambda\Psi_\Theta(t)\varrho, \quad 0 < \varrho \le 1. \qquad (4.12)$$

where Λ is given by (3.7).

Proof. From Remark 4.8, we have

$$
\begin{aligned}
& |z(t) - (\mathcal{K}z)(t)| \\
&= \Bigg|{}_aI^{\alpha,\rho}\Theta(s)(t) - \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg(\sum_{i=1}^{m}\gamma_{ia}I^{\alpha,\rho}\Theta(s)(\eta_i) \\
&\quad + \sum_{j=1}^{n}\kappa_{ja}I^{\alpha-\beta_j,\rho}\Theta(s)(\xi_j) + \sum_{r=1}^{k}\sigma_{ra}I^{\alpha+\delta_r,\rho}\Theta(s)(\theta_r)\Bigg)\Bigg| \\
&\le {}_aI^{\alpha,\rho}|\Theta(s)|(T) + \frac{1}{|\Omega|}\Bigg(\sum_{i=1}^{m}|\gamma_i|_aI^{\alpha,\rho}|\Theta(s)|(\eta_i) \\
&\quad + \sum_{j=1}^{n}|\kappa_j|_aI^{\alpha-\beta_j,\rho}|\Theta(s)|(\xi_j) + \sum_{r=1}^{k}|\sigma_r|_aI^{\alpha+\delta_r,\rho}|\Theta(s)|(\theta_r)\Bigg) \\
&\le \Bigg[\frac{(T-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} + \frac{1}{|\Omega|}\Bigg(\sum_{i=1}^{m}\frac{|\gamma_i|(\eta_i-a)^\alpha}{\rho^\alpha\Gamma(\alpha+1)} \\
&\quad + \sum_{j=1}^{n}\frac{|\kappa_j|(\xi_j-a)^{\alpha-\beta_j}}{\rho^{\alpha-\beta_j}\Gamma(\alpha-\beta_j+1)} + \sum_{r=1}^{k}\frac{|\sigma_r|(\theta_r-a)^{\alpha+\delta_r}}{\rho^{\alpha+\delta_r}\Gamma(\alpha+\delta_r+1)}\Bigg)\Bigg]\Psi_\Theta(t)\varrho \\
&= \Lambda\Psi_\Theta(t)\varrho,
\end{aligned}
$$

□

Next, we show the $\mathcal{UHR}$ and $\mathcal{GUHR}$ stability results.

Theorem 4.10. Suppose that $f_1 : [a,T] \times \mathbb{R}^3 \to \mathbb{R}$ is a continuous function. If (A_2) is satisfied with

$$\frac{2\mathbb{L}_1\Lambda}{1-\mathbb{L}_2} < 1.$$

then the problem (1.1) is $\mathcal{UHR}$ stable as well as $\mathcal{GUHR}$ stable on $[a,T]$.

Proof. Let $z \in \mathbb{E}^1$ is a solution of the inequality (4.4) and u is the unique solution of the problem (1.1). By using the triangle inequality and Lemma 4.6 with (4.11), we have

$$\begin{aligned} |z(t)-u(t)| &= \Bigg| z(t) - {}_aI^{\alpha,\rho}F_u(s)(t) - \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg(A - \sum_{i=1}^{m}\gamma_{ia}I^{\alpha,\rho}F_u(s)(\eta_i) \\ &\quad - \sum_{j=1}^{n}\kappa_{ja}I^{\alpha-\beta_j,\rho}F_u(s)(\xi_j) - \sum_{r=1}^{k}\sigma_{ra}I^{\alpha+\delta_r,\rho}F_u(s)(\theta_r)\Bigg)\Bigg| \\ &= |z(t) - (\mathcal{K}z)(t) + (\mathcal{K}z)(t) - (\mathcal{K}u)(t)| \\ &\leq |z(t) - (\mathcal{K}z)(t)| + |(\mathcal{K}z)(t) - (\mathcal{K}u)(t)| \\ &\leq \Lambda\Psi_\Theta(t)\varrho + \frac{2\mathbb{L}_1\Lambda}{1-\mathbb{L}_2}|z(t)-u(t)|, \end{aligned}$$

where Λ is defined by (3.7), which implies that

$$|z(t)-u(t)| \leq \frac{\Lambda\Psi_\Theta(t)\varrho}{1-\frac{2\mathbb{L}_1\Lambda}{1-\mathbb{L}_2}}.$$

By setting

$$C_{f_1,\Phi} := \frac{\Lambda}{1-\frac{2\mathbb{L}_1\Lambda}{1-\mathbb{L}_2}},$$

we get the following inequality

$$|z(t)-u(t)| \leq C_{f_1,\Phi}\varrho\Psi_\Theta(t).$$

Thus, the problem (1.1) is $\mathcal{UHR}$ stable. Furthermore, if we set $\Phi_f(t) = \varrho\Psi_\Theta(t)$, with $\Phi_f(0) = 0$, so that the problem (1.1) is $\mathcal{GUHR}$ stable. The proof is completed. $\square$

4.2 Stability Results for Proportional Fractional Langevin Differential Equation.

In this section, we propose the Ulam stability for the proportional fractional Langevin differential equation with non-local fractional integral conditions in problem (1.2), that is $\mathcal{UH}$ stable, $\mathcal{GUH}$ stable, $\mathcal{UHR}$ stable and $\mathcal{GUHR}$ stable.

Definition 4.11. The problem (1.2) is called $\mathcal{UH}$ stable, if there is a constant $\Phi > 0$, for every $\varrho > 0$ and for each $w \in \mathbb{E}^1 = C^1([a,T],\mathbb{R})$ a solution of the inequality

$$\left|{}_a^C D^{q_1,\rho,v}\left({}_a^C D^{q_2,\rho,v} + \lambda(t)\right)w(t) - f_2(t, w(t), w(\theta(t)), (\mathcal{K}w)(t))\right| \leq \varrho, \quad t \in [a,T], \tag{4.13}$$

there is a solution $y \in \mathbb{E}^1$ of the problem (1.2) such that

$$|w(t) - y(t)| \leq \Phi\varrho, \quad t \in [a,T]. \tag{4.14}$$

Definition 4.12. The problem (1.2) is called $\mathcal{GUH}$ stable, if there is $\Phi_{f_2} \in C(\mathbb{R}^+,\mathbb{R}^+)$ with $\Phi_{f_2}(0) = 0$ such that for each $w \in \mathbb{E}^1$ a solution of inequality (4.13) there is a solution $y \in \mathbb{E}^1$ of the problem (1.2) such that

$$|w(t) - y(t)| \leq \Phi_{f_2}\varrho, \quad t \in [a,T]. \tag{4.15}$$

Definition 4.13. The problem (1.2) is called be $\mathcal{UHR}$ stable with respect to $\Phi_{f_2} \in C([a,T],\mathbb{R}^+)$, if there is a real number $C_{f_2,\Phi} > 0$ such that for every $\varrho > 0$ and for each $w \in \mathbb{E}^1$ a solution of the inequality

$$\left|{}_a^C D^{q_1,\rho,v}\left({}_a^C D^{q_2,\rho,v} + \lambda(t)\right)w(t) - f_2(t, w(t), w(\theta(t)), (\mathcal{K}w)(t))\right| \leq \varrho\Phi_{f_2}(t), \quad t \in [a,T], \tag{4.16}$$

there is a solution $y \in \mathbb{E}^1$ of the problem (1.2) such that

$$|w(t) - y(t)| \leq C_{f_2,\Phi}\varrho\Phi_{f_2}(t), \quad t \in [a,T]. \tag{4.17}$$

Definition 4.14. The problem (1.2) is called $\mathcal{GUHR}$ stable with respect to $\Phi_{f_2} \in C([a,T],\mathbb{R}^+)$, if there is a real number $C_{f_2,\Phi} > 0$ such that for each $w \in \mathbb{E}^1$ a solution of the inequality

$$\left|{}_a^C D^{q_1,\rho,v}\left({}_a^C D^{q_2,\rho,v} + \lambda(t)\right)w(t) - f_2(t, w(t), w(\theta(t)), (\mathcal{K}w)(t))\right| \leq \Phi_{f_2}(t), \quad t \in [a,T], \tag{4.18}$$

there is a solution $y \in \mathbb{E}^1$ of the problem (1.2) such that

$$|w(t) - y(t)| \le C_{f_2,\Phi}\Phi_{f_2}(t), \quad t \in [a, T]. \tag{4.19}$$

Remark 4.15. A function $w \in \mathbb{E}^1$ is a solution of the inequality (4.13) iff there is a function $\Psi \in C([a, T], \mathbb{R})$, such that

(i) $|\Psi(t)| \le \varrho, \forall t \in [a, T]$.

(ii) ${}_a^C D^{q_1,\rho}\big({}_a^C D^{q_2,\rho} + \lambda(t)\big)w(t) = f_2(t, w(t), w(\theta(t)), (\mathcal{K}w)(t)) + \Psi(t), \quad t \in [a, T]$.

By Remark 4.15, the solution of the equation

$${}_a^C D^{q_1,\rho}\big({}_a^C D^{q_2,\rho} + \lambda(t)\big)w(t) = f_2(t, w(t), w(\theta(t)), (\mathcal{K}w)(t)) + \Psi(t), \quad t \in [a, T],$$

can be written as

$$\begin{aligned}
&w(t)\\
=\;& {}_aI^{q_1+q_2,\rho}F_w(s)(t) - {}_aI^{q_2,\rho}\lambda(s)w(s)(t)\\
&+\frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right)\\
&\times\Bigg(\sum_{j=1}^{n}\alpha_j\Big[{}_aI^{q_1+q_2+\beta_j,\rho}F_w(s)(\eta_j) - {}_aI^{q_2+\beta_j,\rho}\lambda(s)w(s)(\eta_j)\Big]\\
&-\sum_{i=1}^{m}\kappa_i\Big[{}_aI^{q_1+q_2+\mu_i,\rho}F_w(s)(\sigma_i) - {}_aI^{q_2+\mu_i,\rho}\lambda(s)w(s)(\sigma_i)\Big]\Bigg)\\
&+\left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\\
&\times\Bigg(\sum_{l=1}^{r}\nu_l\left[{}_aI^{q_1+q_2+\varphi_l,\rho}F_w(s)(\xi_l) - {}_aI^{q_2+\varphi_l,\rho}\lambda(s)w(s)(\xi_l)\right]\\
&-\sum_{k=1}^{p}\omega_k\left[{}_aI^{q_1+q_2+\gamma_k,\rho}F_w(s)(\psi_k) - {}_aI^{q_2+\gamma_k,\rho}\lambda(s)w(s)(\psi_k)\right]\Bigg)\Bigg]\\
&+{}_aI^{q_1+q_2,\rho}\Psi(s)(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right)\\
&\left(\sum_{j=1}^{n}\alpha_{j\,a}I^{q_1+q_2+\beta_j,\rho}\Psi(s)(\eta_j) - \sum_{i=1}^{m}\kappa_{i\,a}I^{q_1+q_2+\mu_i,\rho}\Psi(s)(\sigma_i)\right)\\
&+\left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\\
&\times\left(\sum_{l=1}^{r}\nu_{l\,a}I^{q_1+q_2+\varphi_l,\rho}\Psi(s)(\xi_l) - \sum_{k=1}^{p}\omega_{k\,a}I^{q_1+q_2+\gamma_k,\rho}\Psi(s)(\psi_k)\right)\Bigg].
\end{aligned} \tag{4.20}$$

First of all, we propose a major Lemma that will be used in the proofs of the stability theorem.

Lemma 4.16. If $w \in \mathbb{E}^1$ satisfies the inequality (4.13), then the function w is a solution of the following inequality

$$|w(t) - (\mathcal{Q}w)(t)| \leq \Lambda_3(q_1+q_2)\varrho, \quad 0 < \varrho \leq 1, \tag{4.21}$$

where $\Lambda_3(q_1+q_2)$ is given by (3.25).

Proof. From Remark 4.15 with (4.20), we obtain

$$\begin{aligned}
& |w(t) - (\mathcal{Q}w)(t)| \\
= & \Bigg| {}_aI^{q_1+q_2,\rho}\Psi(s)(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right) \\
& \times\left(\sum_{j=1}^{n}\alpha_j {}_aI^{q_1+q_2+\beta_j,\rho}\Psi(s)(\eta_j) - \sum_{i=1}^{m}\kappa_i {}_aI^{q_1+q_2+\mu_i,\rho}\Psi(s)(\sigma_i)\right) \\
& +\left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right) \\
& \times\left(\sum_{l=1}^{r}\nu_l {}_aI^{q_1+q_2+\varphi_l,\rho}\Psi(s)(\xi_l) - \sum_{k=1}^{p}\omega_k {}_aI^{q_1+q_2+\gamma_k,\rho}\Psi(s)(\psi_k)\right)\Bigg]\Bigg| \\
\leq & \Bigg\{\frac{(T-a)^{q_1+q_2}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)} \\
& +\Lambda_1\left(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)} + \sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\right) \\
& +\Lambda_2\left(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)} + \sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\right)\Bigg\}\varrho \\
= & \Lambda_3(q_1+q_2)\varrho,
\end{aligned}$$

□

Now, we propose the $\mathcal{UH}$ stability and $\mathcal{GUH}$ stability results.

Theorem 4.17. Assume that (H_1), (H_2), (H_3), (H_4) are satisfied with

$$2L_1\Lambda_3(q_1+q_2) + L_2\phi_0\Lambda_3(q_1+q_2+1) + \Lambda_4 < 1.$$

where $\Lambda_3(u)$, $u \in \{q_1+q_2, q_1+q_2+1\}$, Λ_4 are defined by (3.25) and (3.26), respectively. Then the problem (1.2) is both $\mathcal{UH}$ stable and $\mathcal{GUH}$ stable on $[a,T]$.

Proof. Let $w \in \mathbb{E}^1$ be a solution of the inequality (4.13) and y be the unique solution of the problem (1.2),

By using the triangle inequality, and Lemma 4.16, we have

$$
\begin{aligned}
& |w(t)-y(t)| \\
= & \Bigg| w(t) - \Bigg\{ {}_aI^{q_1+q_2,\rho}F_y(s)(t) - {}_aI^{q_2,\rho}\lambda(s)y(s)(t) \\
& + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right) \\
& \times \Bigg(\sum_{j=1}^{n}\alpha_j\left[{}_aI^{q_1+q_2+\beta_j,\rho}F_y(s)(\eta_j) - {}_aI^{q_2+\beta_j,\rho}\lambda(s)y(s)(\eta_j)\right] \\
& - \sum_{i=1}^{m}\kappa_i\left[{}_aI^{q_1+q_2+\mu_i,\rho}F_y(s)(\sigma_i) - {}_aI^{q_2+\mu_i,\rho}\lambda(s)y(s)(\sigma_i)\right]\Bigg) \\
& + \left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right) \\
& \times \Bigg(\sum_{l=1}^{r}\nu_l\left[{}_aI^{q_1+q_2+\varphi_l,\rho}F_y(s)(\xi_l) - {}_aI^{q_2+\varphi_l,\rho}\lambda(s)y(s)(\xi_l)\right] \\
& - \sum_{k=1}^{p}\omega_k\left[{}_aI^{q_1+q_2+\gamma_k,\rho}F_y(s)(\psi_k) - {}_aI^{q_2+\gamma_k,\rho}\lambda(s)y(s)(\psi_k)\right]\Bigg)\Bigg]\Bigg\}\Bigg| \\
= & |w(t)-(\mathcal{Q}w)(t)+(\mathcal{Q}w)(t)-(\mathcal{Q}y)(t)| \\
\leq & |w(t)-(\mathcal{Q}w)(t)| + |(\mathcal{Q}w)(t)-(\mathcal{Q}y)(t)| \\
\leq & \Lambda_3(q_1+q_2)\varrho + [2L_1\Lambda_3(q_1+q_2) + L_2\phi_0\Lambda_3(q_1+q_2+1) + \Lambda_4]\,|w(t)-y(t)|,
\end{aligned}
$$

So we have

$$|w(t)-y(t)| \leq \frac{\Lambda_3(q_1+q_2)\varrho}{1-[2L_1\Lambda_3(q_1+q_2) + L_2\phi_0\Lambda_3(q_1+q_2+1) + \Lambda_4]}.$$

And setting

$$\Phi := \frac{\Lambda_3(q_1+q_2)}{1-[2L_1\Lambda_3(q_1+q_2) + L_2\phi_0\Lambda_3(q_1+q_2+1) + \Lambda_4]}, \tag{4.22}$$

we have

$$|w(t)-y(t)| \leq \Phi\varrho.$$

Hence, the problem (1.2) is $\mathcal{UH}$ stable. Furthermore, if we set $\Phi_{f_2}(\varrho) = \Phi\varrho$ with $\Phi_{f_2}(0) = 0$, then the problem (1.2) is $\mathcal{GUH}$ stable. The proof is completed. □

Remark 4.18. A function $w \in \mathbb{E}^1$ is a solution of the inequality (4.16) if f there is a function $\Theta \in C([a,T],\mathbb{R})$, such that

(i) $|\Theta(t)| \leq \varrho \Psi_\Theta(t), \forall t \in [a,T]$.

(ii) ${}_a^CD^{q_1,\rho}\big({}_a^CD^{q_2,\rho} + \lambda(t)\big)w(t) = f_2(t, w(t), w(\theta(t)), (\mathcal{K}w)(t)) + \Theta(t), \quad t \in [a,T]$.

By Remark 4.18, the solution of the equation

$${}_a^CD^{\beta,\rho}\big({}_a^CD^{\alpha,\rho} + \lambda(t)\big)w(t) = f_2(t, w(t), w(\theta(t)), (\mathcal{K}w)(t)) + \Theta(t), \quad t \in [a,T],$$

can be written as

$$\begin{aligned}
&w(t)\\
=\ & {}_aI^{q_1+q_2,\rho}F_w(s)(t) - {}_aI^{q_2,\rho}\lambda(s)w(s)(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right)\\
&\times\Bigg(\sum_{j=1}^{n}\alpha_j\Big[{}_aI^{q_1+q_2+\beta_j,\rho}F_w(s)(\eta_j) - {}_aI^{q_2+\beta_j,\rho}\lambda(s)w(s)(\eta_j)\Big]\\
&-\sum_{i=1}^{m}\kappa_i\Big[{}_aI^{q_1+q_2+\mu_i,\rho}F_w(s)(\sigma_i) - {}_aI^{q_2+\mu_i,\rho}\lambda(s)w(s)(\sigma_i)\Big]\Bigg)\\
&+\left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\\
&\times\Bigg(\sum_{l=1}^{r}\nu_l\left[{}_aI^{q_1+q_2+\varphi_l,\rho}F_w(s)(\xi_l) - {}_aI^{q_2+\varphi_l,\rho}\lambda(s)w(s)(\xi_l)\right]\\
&-\sum_{k=1}^{p}\omega_k\left[{}_aI^{q_1+q_2+\gamma_k,\rho}F_w(s)(\psi_k) - {}_aI^{q_2+\gamma_k,\rho}\lambda(s)w(s)(\psi_k)\right]\Bigg)\Bigg]\\
&+{}_aI^{q_1+q_2,\rho}\Theta(s)(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right)\\
&\times\left(\sum_{j=1}^{n}\alpha_{j\,a}I^{q_1+q_2+\beta_j,\rho}\Theta(s)(\eta_j) - \sum_{i=1}^{m}\kappa_{i\,a}I^{q_1+q_2+\mu_i,\rho}\Theta(s)(\sigma_i)\right)\\
&+\left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right)\\
&\times\left(\sum_{l=1}^{r}\nu_{l\,a}I^{q_1+q_2+\varphi_l,\rho}\Theta(s)(\xi_l) - \sum_{k=1}^{m}\omega_{k\,a}I^{q_1+q_2+\gamma_k,\rho}\Theta(s)(\psi_k)\right)\Bigg].
\end{aligned} \tag{4.23}$$

Lemma 4.19. Let $w \in \mathbb{E}^1$ be a solution of inequality (4.16). Then the function w satisfies the inequality

$$|w(t) - (\mathcal{Q}w)(t)| \leq \Lambda_3(q_1+q_2)\Psi_\Theta(t)\varrho, \quad 0 < \varrho \leq 1, \tag{4.24}$$

where $\Lambda_3(q_1+q_2)$ is given by (3.25).

Proof. From Remark 4.18, we get

$$
\begin{aligned}
& |w(t)-(\mathcal{Q}w)(t)| \\
= & \Bigg| {}_aI^{q_1+q_2,\rho}\Psi_\Theta(s)(t) + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right) \\
& \times\left(\sum_{j=1}^{n}\alpha_{ja}I^{q_1+q_2+\beta_j,\rho}\Psi_\Theta(s)(\eta_j) - \sum_{i=1}^{m}\kappa_{ia}I^{q_1+q_2+\mu_i,\rho}\Psi_\Theta(s)(\sigma_i)\right) \\
& +\left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right) \\
& \times\left(\sum_{l=1}^{r}\nu_{la}I^{q_1+q_2+\varphi_l,\rho}\Psi_\Theta(s)(\xi_l) - \sum_{k=1}^{p}\omega_{ka}I^{q_1+q_2+\gamma_k,\rho}\Psi_\Theta(s)(\psi_k)\right)\Bigg]\Bigg| \\
\leq & \Bigg\{\frac{(T-a)^{q_1+q_2}}{\rho^{q_1+q_2}\Gamma(q_1+q_2+1)} \\
& +\Lambda_1\left(\sum_{j=1}^{n}\frac{|\alpha_j|(\eta_j-a)^{q_1+q_2+\beta_j}}{\rho^{q_1+q_2+\beta_j}\Gamma(q_1+q_2+\beta_j+1)} + \sum_{i=1}^{m}\frac{|\kappa_i|(\sigma_i-a)^{q_1+q_2+\mu_i}}{\rho^{q_1+q_2+\mu_i}\Gamma(q_1+q_2+\mu_i+1)}\right) \\
& +\Lambda_2\left(\sum_{l=1}^{r}\frac{|\nu_l|(\xi_l-a)^{q_1+q_2+\varphi_l}}{\rho^{q_1+q_2+\varphi_l}\Gamma(q_1+q_2+\varphi_l+1)} + \sum_{k=1}^{p}\frac{|\omega_k|(\psi_k-a)^{q_1+q_2+\gamma_k}}{\rho^{q_1+q_2+\gamma_k}\Gamma(q_1+q_2+\gamma_k+1)}\right)\Bigg\} \\
& \times\Psi_\Theta(t)\varrho \\
= & \Lambda_3(q_1+q_2)\Psi_\Theta(t)\varrho,
\end{aligned}
$$

□

Next, we are ready to prove $\mathcal{UHR}$ and $\mathcal{GUHR}$ stability results.

Theorem 4.20. Assume that (H_1), (H_2), (H_3), (H_4) are holded with

$$2L_1\Lambda_3(q_1+q_2) + L_2\phi_0\Lambda_3(q_1+q_2+1) + \Lambda_4 < 1.$$

where $\Lambda_3(u)$, $u \in \{q_1+q_2, q_1+q_2+1\}$, Λ_4 are defined by (3.25) and (3.26), respectively. Then the problem (1.2) is both $\mathcal{UHR}$ stable and $\mathcal{GUHR}$ stable on $[a,T]$.

Proof. Let $w \in \mathbb{E}^1$ be a solution of the inequality (4.16) and y be the unique solution of the problem (1.2). By applying the triangle inequality and Lemma 4.19 with (4.23), we get

$$
\begin{aligned}
& |w(t)-y(t)| \\
= & \Bigg| w(t) - \Bigg\{ {}_aI^{q_1+q_2,\rho}F_y(s)(t) - {}_aI^{q_2,\rho}\lambda(s)y(s)(t) \\
& + \frac{e^{\frac{\rho-1}{\rho}(t-a)}}{\Omega}\Bigg[\left(\frac{\Omega_4(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)} - \Omega_3\right) \\
& \times \left(\sum_{j=1}^{n}\alpha_j\left[{}_aI^{q_1+q_2+\beta_j,\rho}F_y(s)(\eta_j) - {}_aI^{q_2+\beta_j,\rho}\lambda(s)y(s)(\eta_j)\right]\right. \\
& \left. - \sum_{i=1}^{m}\kappa_i\left[{}_aI^{q_1+q_2+\mu_i,\rho}F_y(s)(\sigma_i) - {}_aI^{q_2+\mu_i,\rho}\lambda(s)y(s)(\sigma_i)\right]\right) \\
& + \left(\Omega_1 - \frac{\Omega_2(t-a)^{q_2}}{\rho^{q_2}\Gamma(q_2+1)}\right) \\
& \times \left(\sum_{l=1}^{r}\nu_l\left[{}_aI^{q_1+q_2+\varphi_l,\rho}F_y(s)(\xi_l) - {}_aI^{q_2+\varphi_l,\rho}\lambda(s)y(s)(\xi_l)\right]\right. \\
& \left. - \sum_{k=1}^{p}\omega_k\left[{}_aI^{q_1+q_2+\gamma_k,\rho}F_y(s)(\psi_k) - {}_aI^{q_2+\gamma_k,\rho}\lambda(s)y(s)(\psi_k)\right]\right)\Bigg]\Bigg\}\Bigg| \\
= & |w(t) - (\mathcal{Q}w)(t) + (\mathcal{Q}w)(t) - (\mathcal{Q}y)(t)| \\
\le & |w(t) - (\mathcal{Q}w)(t)| + |(\mathcal{Q}w)(t) - (\mathcal{Q}y)(t)| \\
\le & \Lambda_3(q_1+q_2)\Psi_\Theta(t)\varrho + \left[2L_1\Lambda_3(q_1+q_2) + L_2\phi_0\Lambda_3(q_1+q_2+1) + \Lambda_4\right]|w(t)-y(t)|,
\end{aligned}
$$

implies that

$$|w(t)-y(t)| \le \frac{\Lambda_3(q_1+q_2)\Psi_\Theta(t)\varrho}{1-\left[2L_1\Lambda_3(q_1+q_2) + L_2\phi_0\Lambda_3(q_1+q_2+1) + \Lambda_4\right]}.$$

By setting

$$C_{f,\Phi} := \frac{\Lambda_3(q_1+q_2)}{1-\left[2L_1\Lambda_3(q_1+q_2) + L_2\phi_0\Lambda_3(q_1+q_2+1) + \Lambda_4\right]},$$

we get the following inequality

$$|z(t)-x(t)| \le C_{f,\Phi}\varrho\Psi_\Theta(t).$$

Hence, the problem (1.2) is $\mathcal{UHR}$ stable. Furthermore, if we set $\Phi_f(t) = \varrho\Psi_\Theta(t)$, with $\Phi_f(0) = 0$, then the problem (1.2) is $\mathcal{GUHR}$ stable. The proof is completed. □

4.3 Examples

In this section, we present the some examples to support the main results for the problem (1.1) and problem (1.2)

4.3.1 Example for pantograph equation

Example 4.21. Consider the following nonlinear GPF pantograph differential equation via mixed nonlocal conditions of the form:

$$\begin{cases} {}^{C}D^{\frac{2}{3},\frac{1}{2}}u(t) = \dfrac{2+|u(t)|+|u(\frac{3}{2}t)|+|{}^{C}D^{\frac{2}{3},\frac{1}{2}}u(\frac{3}{2}t)|}{95e^{2t}\cos 2t\left(1+|u(t)|+|u(\frac{3}{2}t)|+|{}^{C}D^{\frac{2}{3},\frac{1}{2}}u(\frac{3}{2}t)|\right)}, \quad t\in[0,2], \quad , \\ \displaystyle\sum_{i=1}^{2}\left(\frac{i+1}{2}\right)u\left(\frac{2i+1}{3}\right)+\sum_{j=1}^{3}\left(\frac{2j-1}{5}\right){}^{C}D^{\frac{2j+1}{10},\frac{1}{2}}u\left(\frac{j}{2}\right) \\ \displaystyle\qquad +\sum_{r=1}^{2}\left(\frac{r}{3}\right)I^{\frac{r}{r+1},\frac{1}{2}}u\left(\frac{r+1}{2r}\right)=1, \end{cases} \tag{4.25}$$

Here $\alpha = 2/3$, $\rho = 1/2$, $\lambda = 3/2$, $a = 0$, $T = 2$, $m = 2$, $n = 3$, $k = 2$, $\gamma_i = (i+1)/2$, $\eta_i = (2i+1)/3$, $i = 1,2$, $\kappa_j = (2j-1)/5$, $\beta_j = (2j+1)/10$, $\xi_j = j/2$, $j = 1,2$, $\sigma_r = r/3$, $\delta_r = r/(r+1)$, $\theta_r = (r+1)/2r$, $r = 1,2$, From the given all datas, we obtain that $\Omega \approx 1.3039822 \neq 0$, and $\Lambda_1 \approx 9.7044$. and

$$f(t,u,v,w) = \frac{2+|u|+|v|+|w|}{95e^{2t}\cos 2t\,(1+|u|+|v|+|w|)}.$$

For $u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{R}$ and $t \in [0,2]$, we have

$$|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)| \leq \frac{1}{95e^{2t}\cos 2t}\left(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|\right).$$

The assumptions (H_1) is satisfied with $L_1 = L_2 = \frac{1}{95}$. Hence

$$\frac{2L_1\Lambda_1}{1-L_2} \approx 0.206476 < 1.$$

Since, all the assumptions of Theorem 3.2 are satisfied, then the problem (4.25) has a unique solution on $[0,2]$. Furthermore, we can also compute that

$$\Phi := \frac{\Lambda_1}{1-\frac{2L_1\Lambda_1}{1-L_2}} \approx 12.22949 > 0.$$

Hence, by Theorem 4.7, the problem (4.25) is both $\mathcal{UH}$ and also $\mathcal{GUH}$ stable.

Example 4.22. Consider the following nonlinear GPF pantograph differential equation via mixed nonlocal conditions of the form:

$$\begin{cases} {}^{C}D^{\frac{1}{2},\frac{1}{3}}u(t) = \dfrac{1}{4^{t+3}\left(1+|u(t)|+|u(\frac{1}{6}t)|+|{}^{C}D^{\frac{1}{2},\frac{1}{3}}u(\frac{1}{6}t)|\right)}, \quad t\in(0,2], \\ \displaystyle\sum_{i=1}^{3}\left(\frac{i+1}{2i}\right)u\left(\frac{i+1}{4}\right)+\sum_{j=1}^{2}\left(\frac{2j-1}{3}\right){}^{C}D^{\frac{j}{j+1},\frac{1}{2}}u\left(\frac{j}{2}\right) \\ \displaystyle\qquad +\sum_{r=1}^{2}\left(\frac{r-1}{3}\right)I^{\frac{2r}{r+1},\frac{1}{2}}u\left(\frac{r+1}{2r}\right)=5, \end{cases} \tag{4.26}$$

Here $\alpha = 1/2$, $\rho = 1/3$, $\lambda = 1/6$, $a = 0$, $T = 2$, $m = 3$, $n = 2$, $k = 2$, $\gamma_i = (i+1)/2i$, $\eta_i = (i+1)/4$, $i = 1,2,3$, $\kappa_j = (2j-1)/3$, $\beta_j = j/(j+1)$, $\xi_j = j/2$, $j = 1,2$, $\sigma_r = (r-1)/3$, $\delta_r = 2r/(r+1)$, $\theta_r = (r+1)/2r$, $r = 1,2$, From the given all datas, we obtain that $\Omega \approx 0.809627 \neq 0$, and $\Lambda_1 \approx 10.02678$. and

$$f(t,u,v,w) = \frac{1}{4^{t+3}\left(1+|u|+|v|+|w|\right)}.$$

For u_1, u_2, v_1, v_2, w_1, $w_2 \in \mathbb{R}$ and $t \in [0,2]$, we have

$$|f(t,u_1,v_1,w_1)-f(t,u_2,v_2,w_2)| \le \frac{1}{4^{t+3}}\left(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|\right).$$

The assumptions (H_1) is satisfied with $L_1 = L_2 = \frac{1}{64}$. Hence

$$\frac{2L_1\Lambda_1}{1-L_2} \approx 0.31831 < 1.$$

we can also compute that

$$\Phi := \frac{\Lambda_1}{1-\frac{2L_1\Lambda_1}{1-L_2}} \approx 14.708709 > 0.$$

Hence, by Theorem 4.10, the problem (4.26) is both $\mathcal{UHR}$ and also $\mathcal{GUHR}$ stable.

4.3.2 Example for Langevin equation

Example 4.23. Consider the following nonlinear GPF functional integro-differential Langevin equation involving nonlocal integral conditions of the form:

$$\begin{cases} {}^C_a D^{\frac{3}{4},\frac{1}{3}} \left({}^C_a D^{\frac{1}{2},\frac{1}{3}} + \dfrac{1}{25}(t-a)^2 e^{\frac{\rho-1}{\rho}(t-a)} \right) y(t) = f(t, y(t), y(\theta(t)), (\mathcal{K}y)(t)), \quad t \in (0,2] \\ \displaystyle\sum_{i=1}^{3} \left(\frac{i}{3}\right) {}_aI^{\frac{i}{i+1},\frac{1}{3}} y\left(\frac{i}{2(i+1)}\right) = \sum_{j=1}^{2} \left(\frac{j}{4}\right) {}_aI^{\frac{j+1}{j+2},\frac{1}{3}} y\left(\frac{j}{j+6}\right), \\ \displaystyle\sum_{k=1}^{2} \left(\frac{k}{5}\right) {}_aI^{\frac{k+2}{k+3},\frac{1}{3}} y\left(\frac{\sqrt{k}}{k^2+2}\right) = \sum_{l=1}^{3} \left(\frac{l}{6}\right) {}_aI^{\frac{l+3}{l+4},\frac{1}{3}} y\left(\frac{2l}{3l+2}\right). \end{cases} \tag{4.27}$$

Here $q_1 = \frac{1}{2}$, $q_2 = \frac{1}{2}$, $\rho = \frac{1}{3}$, $a = 0$, $T = 2$, $m = 3$, $n = 2$, $p = 2$, $r = 3$, $\kappa_i = \frac{i}{3}$, $\sigma_i = \frac{i}{2(i+1)}$, $\mu_i = \frac{i}{i+1}$, $i = 1, 2, 3$, $\alpha_j = \frac{j}{4}$, $\eta_j = \frac{j}{j+6}$, $\beta_j = \frac{j+1}{j+2}$, $j = 1, 2$, $\omega_k = \frac{k}{5}$, $\psi_j = \frac{\sqrt{k}}{k^2+2}$, $\gamma_k = \frac{k+2}{k+3}$, $k = 1, 2$, $\nu_l = \frac{l}{6}$, $\xi_l = \frac{2l}{3l+2}$, $\varphi_l = \frac{l+3}{l+4}$, $l = 1, 2, 3$, $\theta(t) = \frac{t}{2}$ and

$$\lambda(t) = \frac{1}{25}(t-a)^2 e^{\frac{\rho-1}{\rho}(t-a)}.$$

Obviously, the function λ satisfies (H_2) for all $t \in [a, T]$. From the given datas, we obtain that $\Omega_1 \approx 0.6995071719$, $\Omega_2 \approx 0.7639237899$, $\Omega_3 \approx -0.3023660189$, $\Omega_4 \approx -0.2312067168$, $\Omega \approx 0.0662301783 \neq 0$. Furthermore, we assume the nonlinearity as follows:

(i) Let $f : [a, T] \times \mathbb{R}^3 \to \mathbb{R}$ be a function which is given by

$$\begin{aligned} f(t, y(t), y(\theta(t)), (\mathcal{K}y)(t)) &= \frac{1}{2} + \frac{t^2}{3} + \frac{2\cos^2(\pi t)}{(t+16)^2} \frac{|y|}{1+|y|} - \frac{y(0.5t)}{(t+16)^2} \\ &\quad + \frac{(t+2)^3}{(8e^t+1)^2} \int_a^t \frac{\cos^2(\pi t)}{(e^{s^2}+1)^2} x(s) ds. \end{aligned}$$

For $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$ and $t \in [a, T]$, we havy

$$\begin{aligned} |f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| &\leq \frac{1}{(t+16)^2} \left(|x_1 - y_1| + |x_2 - y_2|\right) \\ &\quad + \frac{(t+2)^3}{(8e^t+1)^2} |z_1 - z_2|, \\ |\phi(t, s, x_1) - \phi(t, s, y_1)| &\leq \frac{1}{4}|x_1 - y_1|. \end{aligned}$$

The hypotheses (H_1)-(H_4) are satisfied with $L_1 = \frac{1}{256}$, $L_2 = \frac{1}{81}$ and $\phi_0 = \frac{1}{4}$. Hence

$$2L_1\Lambda_3(q_1 + q_2) + L_2\phi_0\Lambda_3(q_1 + q_2 + 1) + \Lambda_4 \approx 0.7833485782 < 1.$$

Since, all the hypotheses of Theorem 3.6 are satisfied, the problem (4.27) has a unique solution on $[0, 2]$. Moreover, we can also compute that

$$\Phi := \frac{\Lambda_3(q_1 + q_2)}{1 - [2L_1\Lambda_3(q_1 + q_2) + L_2\phi_0\Lambda_3(q_1 + q_2 + 1) + \Lambda_4]} \approx 375.8602857 > 0.$$

Hence, by Theorem 4.17, the problem (4.27) is both $\mathcal{UH}$ and also $\mathcal{GUH}$ stable.

(ii) Let $f : [a, T] \times \mathbb{R}^3 \to \mathbb{R}$ be a function which is given by

$$\begin{aligned} f(t, y(t), y(\theta(t)), (\mathcal{K}y)(t)) &= \frac{2e^t}{(t+1)^2} + \frac{e^{-t}}{2(t+9)^2} \cdot \frac{|y|}{2+|y|} \\ &\quad + \frac{1}{(t+9)^2} \cdot \frac{|y(0.75t)|}{|y(0.75t)| + 4} \\ &\quad + \frac{\sin^2(\pi t)}{e^t + 1} \int_a^t \frac{\cos^2(t-s)}{(e^{t-s}+1)^2} y(s)ds. \end{aligned}$$

It is easy to see that, for all $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$ and $t \in [a, T]$, we get

$$\begin{aligned} |f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| &\leq \frac{1}{4(t+9)^2} \left(|x_1 - y_1| + |x_2 - y_2|\right) \\ &\quad + \frac{1}{(e^t+1)^3} \left|z_1 - z_2\right|, \\ |\phi(t, s, x_1) - \phi(t, s, y_1)| &\leq \frac{1}{16}|x_1 - y_1|. \end{aligned}$$

The hypotheses (H_1)-(H_4) are satisfied with $L_1 = \frac{1}{324}$, $L_2 = \frac{1}{8}$ and $\phi_0 = \frac{1}{16}$. Hence

$$2L_1\Lambda_3(q_1 + q_2) + L_2\phi_0\Lambda_3(q_1 + q_2 + 1) + \Lambda_4 \approx 0.7348101092 < 1.$$

Furthermore, for $x, y, z \in \mathbb{R}$ and $t \in [a, T]$, it follows that

$$|f(t, x, y, z)| \leq \frac{2e^t}{(t+1)^2} + \frac{e^{-t}}{4(t+9)^2}|x| + \frac{1}{4(t+9)^2}|y| + \frac{1}{(e^t+1)^3}|z|.$$

The hypothesis (H_5) is also valid with $h_1(t) = \frac{2e^t}{(t+1)^2}$, $h_2(t) = \frac{e^{-t}}{4(t+9)^2}$, $h_3(t) = \frac{1}{4(t+9)^2}$, $h_4(t) = \frac{1}{(e^t+1)^3}$ and $h_1^* = 2$, $h_2^* = h_3^* = \frac{1}{324}$, $h_4^* = \frac{1}{8}$. Therefore, all the hypotheses of Theorem 3.8 are fulfilled, which conclude that the problem (4.27) has at least one solution on $[0, 2]$. Moreover, we obtain

$$C_{f,\Phi} := \frac{\Lambda_3(q_1 + q_2)}{1 - [2L_1\Lambda_3(q_1 + q_2) + L_2\phi_0\Lambda_3(q_1 + q_2 + 1) + \Lambda_4]} \approx 307.0654958 > 0.$$

Hence, by Theorem 4.20, the problem (4.27) is both $\mathcal{UHR}$ and $\mathcal{GUHR}$ stable.

CHAPTER 5

CONCLUSION AND DISCUSSION

In this dissertation, we studied the existence results and stability results of the solution for a class of nonlinear fractional pantograph differential equation with mixed nonlocal boundary conditions of problem (1.1) by using Banach's fixed point theorem, Leray-Schauder's Nonlinear alternative and Krasnoselskii's fixed point theorem and integro-differential Langevin equation with nonlocal fractional integral conditions of problem (1.2) by using Banach's fixed point theorem, Krasnoselskii's fixed point theorem and Schaefer's fixed point theorem we investigated sufficient conditions for the existence and uniqueness of solutions of the purpose problem (1.1) and problem (1.2). Moreover, we proved four different types of Ulam stability results including $\mathcal{UH}$ stability, $\mathcal{GUH}$ stability, $\mathcal{UHR}$ stability and $\mathcal{GUHR}$ stability for the problem (1.1) and problem (1.2) . For the justification, numerical examples were given to illustrate our main theoretical results.

Future research

1. Applying the proportional fractional calculus with another problems.
2. Applying Phi-Hilfer, Katugampola, Hadamard,Conformable fractional calculus wiht this problem.

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