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NEO BALANCING NUMBERS

NATDANAI CHAILANGKA

A THESIS PROPOSAL SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR MASTER OF SCIENCE
IN MATHEMATICS
FACULTY OF SCIENCE
BURAPHA UNIVERSITY
2020

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The Thesis of Natdanai Chailangka has been approved by the examining committee to be partial fulfillment of the requirements for the Master of Science in Mathematics of Burapha University

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If a positive integer $n \geq 2$ is a solution of the equation

$$1 + 2 + 3 + \cdots + (n-1) = (n-1) + (n-0) + (n+1) + (n+2) + \cdots + (n+r)$$

for some integer r, n is called a neo balancing number and r is called a neo balancer corresponding to neo balancing number n. The purpose of this paper is to establish a generating function of neo balancing numbers, recurrence relations for neo balancing numbers and an application of neo balancing numbers to a Diophantine equation. Moreover, we prove the relations between neo balancing numbers and balancing numbers.

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Notation

$\mathbb{Z}$	Set of all integers.
$\mathbb{Z}^+$	Set of all positive integers.
B_n	The n^{th} balancing numbers.
C_n	The n^{th} Lucas balancing numbers.
b_n	The n^{th} cobalancing numbers.
F_n	The n^{th} Fibonacci numbers.
L_n	The n^{th} Lucas numbers.
i	The imaginary unit.
P_n	The n^{th} neo balancing numbers.
Q_n	The n^{th} Lucas neo balancing numbers.

CHAPTER 1

INTRODUCTION

In this chapter, we will introduce some background on balancing numbers and some properties of balancing numbers.

Introduction

The definition of balancing numbers was introduced by Behera and Panda (1999). An integer $n \in \mathbb{Z}^+$ is called a balancing number if n is a solution of

$$1 + 2 + 3 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \tag{1.1}$$

for some $r \in \mathbb{Z}^+$. Here r is called the balancer corresponding to the balancing number n. For example,

$$n = 6, r = 2$$

$$1 + 2 + 3 + 4 + (6-1) = (6+1) + (6+2)$$

$$n = 35, r = 14$$

$$1 + 2 + 3 + \cdots + (35-1) = (35+1) + (35+2) + (35+3) + \cdots + (35+14).$$

Then, they found that n is a balancing number if and only if n^2 is a triangular number. Also n is a balancing number if and only if $8n^2 + 1$ is a perfect square. In addition, they found the generating function of balancing numbers, the non-linear first order recurrence, the second order linear recurrence, recurrence relations for balancing numbers, nonrecursive form for balancing numbers and an application of balancing numbers to a Diophantine equation.

Panda (2006) established some other interesting arithmetic-type, de-Moivre's-type and trigonometric-type properties of balancing numbers. Panda also established an important property concerning the greatest common divisor of two balancing numbers.

In this thesis, we will consider n which is a positive integer solution of equation

$$1+2+3+\cdots+(n-1) = (n-1)+(n+0)+(n+1)+(n+2)+\cdots+(n+r) \quad (1.2)$$

for some integer r. The number n is called a neo balancing number and r is called the neo balancer corresponding to the neo balancing number n. We expect that they have properties similar to those of balancing numbers. In addition, we expect that there are relationships between balancing numbers and neo balancing numbers.

Research Objectives

1. To study balancing numbers and their properties.
2. To introduce neo balancing numbers.
3. To study properties of neo balancing numbers.
4. To study relationships between balancing numbers and neo balancing numbers.

Scope of the study

In this thesis, we will study neo balancing numbers $n \in \mathbb{Z}^+$ which satisfy the equation

$$1+2+3+\cdots+(n-1) = (n-1)+(n+0)+(n+1)+(n+2)+\cdots+(n+r)$$

for some $r \in \mathbb{Z}^+ \cup \{-1\}$ and study properties of neo balancing numbers. Finally, we will try to study relationships between neo balancing numbers and balancing numbers.

CHAPTER 2

PRELIMINARIES AND LITERATURE REVIEWS

In this chapter, we will introduce important preliminary notes and some literature reviews concerning neo balancing numbers and balancing numbers.

Preliminaries

Triangular numbers

Definition 2.1. *(Garge & Shirali, 2012) Triangular numbers are numbers associated with triangular arrays of dots. The idea is easier to convey using pictures than words; see Figure 2.1. We see from the figure that if* T_n *denotes the* n^{th} *triangular number, then* $T_1 = 1$, $T_2 = T_1 + 2$, $T_3 = T_2 + 3$, $T_4 = T_3 + 4$. *Thus*

$$T_n = T_{n-1} + n$$

leading to:

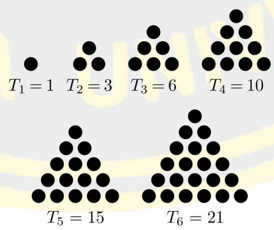

Figure 2.1 The first six triangular numbers.

Diophantine equation

Definition 2.2. *(Sundstrom, 2006) An equation whose solutions are required to be integers is called a Diophantine equation.*

Fibonacci sequence

Definition 2.3. *(Vajda, 2008) The first of two Fibonacci numbers are $F_1 = 1$, $F_2 = 1$, and other terms of the sequence are obtained by means of the recurrence relation*

$$F_{n+1} = F_n + F_{n-1},\ n \geq 2.$$

Lucas sequence

Definition 2.4. *(Vajda, 2008) Lucas Sequence is also obtained from the same recurrence relation as that for Fibonacci numbers. The first two Lucas numbers are $L_1 = 1, L_2 = 3$ and other terms of the sequence are obtained by means of the recurrence relation*

$$L_{n+1} = L_n + L_{n-1},\ n \geq 2.$$

De Moivre's formula

Definition 2.5. *(Alfors, 1979) For any real number x and integer n it holds that*

$$(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$$

where i is the imaginary unit ($i^2 = -1$).

Perfect numbers

Definition 2.6. *(Leinster, 2001) For any positive integer n, define $\sigma(n) = \sum_{d|n} d$, the sum of the positive divisors of n, and call n perfect if $\sigma(n) = 2n$.*

Relavant research

(Behera & Panda, 1999) An integer $n \in \mathbb{Z}^+$ is called a balancing number if n is a solution of

$$1 + 2 + 3 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r) \qquad (2.1)$$

for some $r \in \mathbb{Z}^+$. Here r is called the balancer corresponding to the balancing number n. It follows from equation (2.1) that, if n is a balancing number with balancer r, then

$$n^2 = \frac{(n+r)(n+r+1)}{2} \tag{2.2}$$

and

$$r = \frac{-(2n+1)+\sqrt{8n^2+1}}{2}. \tag{2.3}$$

It is clear from equation (2.2) that we have the following result.

Theorem 2.7. *(Behera & Panda, 1999) For each $n \in \mathbb{Z}^+$, n is a balancing number if and only if n^2 is a triangular number.*

Also, it follows from equation (2.3) that we obtain

Theorem 2.8. *(Behera & Panda, 1999) For each $n \in \mathbb{Z}^+$, n is a balancing number if and only if $8n^2+1$ is a perfect square.*

Behera and Panda also proved many other results as follows.

Theorem 2.9. *(Behera & Panda, 1999) If x is a balancing number, its next balancing number is $3x+\sqrt{8x^2+1}$ and its previous balancing numbers is $3x-\sqrt{8x^2+1}$.*

Lemma 2.10. *(Behera & Panda, 1999) If x is an even balancing number, $3x+\sqrt{8x^2+1}$ is odd.*

Lemma 2.11. *(Behera & Panda, 1999) If x is an odd balancing number, $3x+\sqrt{8x^2+1}$ is even.*

Lemma 2.12. *(Behera & Panda, 1999) If B_n is the n^{th} balancing number, then $B_{n+1} = 3B_n + \sqrt{8B_n^2+1}$ and $B_{n-1} = 3B_n - \sqrt{8B_n^2+1}$.*

Lemma 2.13. *(Behera & Panda, 1999) If B_n is the n^{th} balancing number, then its recurrence relation is $B_{n+1} = 6B_n - B_{n-1}$ when $B_0 = 1$ and $B_1 = 6$.*

Theorem 2.14. *(Behera & Panda, 1999)*

(a) $B_{n+1} \cdot B_{n-1} = (B_n+1)(B_n-1)$.

(b) $B_n = B_k \cdot B_{n-k} - B_{k-1} \cdot B_{n-k-1}$ *for any positive integer* $k < n$.

(c) $B_{2n} = B_n^2 - B_{n-1}^2$.

(d) $B_{2n+1} = B_n(B_{n+1} - B_{n-1})$.

Lemma 2.15. *(Behera & Panda, 1999) If B_n is the n^{th} balancing number, then its Binet form is*

$$B_n = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}; n = 0, 1, 2, ..., \tag{2.4}$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$.

Theorem 2.16. *(Behera & Panda, 1999) The solutions of the Diophantine equation $x^2 + (x+1)^2 = y^2$ are given by*

$$x = \frac{\sqrt{\frac{1}{2}(\sqrt{8B^2+1} - 1) - 1}}{2} \tag{2.5}$$

$$y = \frac{1}{2}\sqrt{1 + \sqrt{8B^2+1}} \tag{2.6}$$

when B is an odd balancing number.

Panda (2006) established some fascinating properties of balancing numbers. Panda and behera defined that 1 is also a balancing number(the reason is that $8 \cdot 1^2 + 1 = 9$ is a perfect square), we can set $B_0 = 1$, $B_1 = 6$, and so on, using the symbol B_n for the n^{th} balancing number. To standardize the notation on par with Fibonacci numbers, Panda relabeled the balacing numbers by setting $B_1 = 1$, $B_2 = 6$ and so on. But we do not relabel the balancing numbers in this thesis. Some results of balancing numbers can be stated with this new convention as follows:

The second order linear recurrence:

$$B_{n+1} = 6B_n - B_{n-1}; n = 2, 3, ... \tag{2.7}$$

The non-linear first order recurrence:

$$B_{n+1} = 3B_n + \sqrt{8B_n^2 + 1}; n = 1, 2, 3, ... \tag{2.8}$$

The relation:

$$B_n = B_{r+1}B_{n-r} - B_rB_{n-r-1}; r = 1, 2, 3, ..., n-2. \tag{2.9}$$

The Binet form:

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}; n = 1, 2, 3, ..., \tag{2.10}$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$.

The interesting relation:

$$B_{n+1}B_{n-1} = (B_n + 1)(B_n - 1). \tag{2.11}$$

Throughout F_n is the n^{th} Fibonacci number, L_n is the n^{th} Lucas number, B_n is the n^{th} Balancing number and $C_n = \sqrt{8B_n^2 + 1}$ where $n \in \mathbb{Z}^+$. Some of the following results suggest that C_n is associated with B_n in the way L_n is associated with F_n. We know that if x and y are complex numbers, then $(x + y)(x - y) = x^2 - y^2$.
In the following result, Panda (2006) obtained an analogous property of balancing numbers.

Theorem 2.17. *(Panda, 2006) If m and n are natural numbers and $m > n$, then*

$$(B_m + B_n)(B_m = B_n) = B_{m+n}B_{m-n}.$$

Remark. *The Fibonacci numbers satisfy a similar property*

$$F_{m+n}F_{m-n} = F_m^2 - (-1)^{m+n}F_n^2.$$

If n is a natural number, then $1+3+\cdots+(2n-1) = n^2$, $2+4+\cdots+2n = n(n+1)$ and $1 + 2 + \cdots + 2n = n(2n + 1)$. In the following result, Panda obtained properties of balancing numbers similar to the above three identities.

Theorem 2.18. *(Panda, 2006) If n is natural number, then*

(a) $B_1 + B_3 + \cdots + B_{2n-1} = B_n^2$
(b) $B_2 + B_4 + \cdots + B_{2n} = B_nB_{n+1}$
(c) $B_1 + B_2 + \cdots + B_{2n} = B_n(B_n + B_{n+1})$.

The complex identity $(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$ is known as the de-Moivre's formula. The following theorem looks like de-Moivre's formula.

Theorem 2.19. *(Panda, 2006) If n and r are natural numbers, then*

$$(C_n + \sqrt{8}B_n)^r = C_{nr} + \sqrt{8}B_{nr}.$$

Remark. *The Fibonacci numbers satisfy a similar property*

$$\left[\frac{L_n + \sqrt{5}F_n}{2}\right]^r = \frac{L_{rn} + \sqrt{5}F_{rn}}{2}.$$

Corollary 2.20. *(Panda, 2006) If n and r are natural numbers, then*

$$(C_n - \sqrt{8}B_n)^r = C_{nr} - \sqrt{8}B_{nr}.$$

The trigonometric identity $\sin{(x+y)} = \sin x \cos y + \cos x \sin y$ is quite well known. In the following theorem, Panda obtained analogous properties of balancing numbers.

Theorem 2.21. *(Panda, 2006) If m and n are natural numbers, then*

$$B_{m+n} = B_m C_n + C_m B_n.$$

Remark. *The corresponding identity for Fibonacci numbers is*

$$F_{m+n} = \frac{1}{2}[F_m L_n + L_m F_n].$$

Corollary 2.22. *(Panda, 2006) If n and r are natural numbers and $m > n$, then*

$$B_{m-n} = B_m C_n - C_m B_n.$$

The following corollary is similar to the trigonometric identity $\sin 2x = 2 \sin x \cos x$.

Corollary 2.23. *(Panda, 2006) If n is a natural numbers, then*

$$B_{2n} = 2B_n C_n.$$

For any two integers m and n, Panda denoted the greatest common divisor of m and n by (m, n). The Fibonacci property F_m divides F_n if and only if m divides n and $(F_m, F_n) = F_{(m,n)}$. In the following theorem, Panda obtained a property of balancing numbers analogous to that of Fibonacci numbers.

Theorem 2.24. *(Panda, 2006) If m and n are natural numbers, then*

$$B_m \textit{ divides } B_n \textit{ if and only if } m \textit{ divides } n.$$

We need the following lemmas to prove theorem 2.24.

Lemma 2.25. *(Panda, 2006) If n is a natural number, then*

$$(B_n, C_n) = 1.$$

Lemma 2.26. *(Panda, 2006) If n and k are natural numbers, then*

$$B_k \textit{ divides } B_{nk}.$$

Lemma 2.27. *(Panda, 2006) If n and k are natural numbers, then*

$$(B_k, C_{nk}) = 1.$$

Lemma 2.28. *(Panda, 2006) If n and k are natural numbers and B_k divides B_n, then*

$$k \textit{ divides } n.$$

Finally, Panda proves the following theorem.

Theorem 2.29. *(Panda, 2006) If m and n are natural numbers, then*

$$(B_m, B_n) = B_{(m,n)}.$$

Literature Reviews

Liptai (2004) studied Fibonacci numbers in balancing numbers. Liptai found that there are no Fibonacci Balancing numbers except 1 using connection with Fibonacci sequence, Lucas sequence and Pell's equation. Later on Panda (2007) gave other results of balancing numbers by proving that the only sequence cobalancing numbers (the definition is given below) in the Fibonacci sequence is $F_2 = 1$.

Panda and Ray (2005) introduced cobalancing numbers and cobalancers in a way similar to the balancing numbers. They call $n \in \mathbb{Z}^+$ a cobalancing number if

$$1 + 2 + 3 + \cdots + n = (n+1) + (n+2) + \cdots + (n+r).$$

for some $r \in \mathbb{Z}^+$. Here, they call r the cobalancer corresponding to the cobalancing number n. The first three cobalancing numbers are 2, 14 and 84 with cobalancers 1, 6 and 35, respectively.

They have 2 theorems that, n is a cobalancing number if and only if $8n^2+8n+1$ is a perfect square, that is, $n(n+1)$ is a triangular number. Since $8(0)^2+8(0)+1=1$ is a perfect square, they count 0 as a cobalancing number. Furthermore, they use the notation b_n for the n^{th} cobalancing number. They set $b_1 = 0$, $b_2 = 2$ and $b_3 = 14$, where n is a positive integer. Then the recurrence relation for cobalancing number is given by $b_{n+1} = 6b_n - b_{n-1} + 2$.

Some results were established by Panda and Ray as follows:

The function generating next cobalancing numbers is given by

$$g(x) = 3x + \sqrt{8x^2+8x+1} + 1 \tag{2.12}$$

when x is a cobalancing number.

Some other interesting relations concerning cobalancing numbers include:

(a) $(b_n - 1)^2 = 1 + b_{n-1}b_{n+1}$.

(b) $b_n = b_k + B_k b_{n-k+1} - B_{k-1}b_{n-k}$ for any positive integer $n > k \geq 2$.

(c) $b_{2n} = B_n b_{n+1} - b_n(B_{n-1} - 1)$.

(d) $b_{2n+1} = (B_{n+1} + 1)b_{n+1} - B_n b_n$.

The Binet form:

$$b_n = \frac{\lambda_1^{n-1/2} - \lambda_2^{n-1/2}}{\lambda_1 - \lambda_2} - \frac{1}{2}; n = 1, 2, ..., \tag{2.13}$$

where $\lambda_1 = 3+\sqrt{8}$, $\lambda_2 = 3-\sqrt{8}$, $\sqrt{\lambda_1} = 1+\sqrt{2}$ and $\sqrt{\lambda_2} = 1-\sqrt{2}$.

An application of cobalancing numbers to a Diophantine equation is as follows.

Panda and Ray considered the solution of

$$x^2 + (x+1)^2 = y^2 \tag{2.14}$$

They let b be any cobalancing number and r its cobalancer and $c = b + r$. Then they have

$$x = b + r = c \tag{2.15}$$

and

$$y = \sqrt{2c^2 + 2c + 1}. \tag{2.16}$$

is a solution to equation (2.14).

Keskin and Karaatly (2012) studied properties of balancing nubmers and square triangular numbers using connection with triangular numbers, Oblong numbers, generalized Fibonacci numbers, generalized Lucas numbers, Pell sequence and Pell-Lucas sequence.

Panda and Davala (2015) studied the perfect numbers in the balancing numbers called perfect balancing numbers. However, 6 is the only perfect number in the balancing numbers using results of Pell and associated Pell numbers and Fibonacci and Lucas numbers including balancing and Lucas balancing numbers.

Gautam (2018) studied the origin of balancing numbers. In addition, he found connection with triangular numbers, Pells numbers and Fibonacci numbers. He discussed the generating functions and recurrence relations which play precious role in searching more balancing numbers from the given balancing numbers.

CHAPTER 3

RESEARCH METHODOLOGY

In this thesis, we will study neo balancing numbers. We do the following process.

1. We study behavior equation

$$1 + 2 + 3 + \cdots + (n-1) = (n-1) + (n+0) + (n+1) + (n+2) + \cdots + (n+r)$$

2. We make conjectures about neo balancing numbers based on known results about balancing numbers.
3. We prove our conjectures and make a conclusion.
4. We establish the some relationship between neo balancing numbers and balancing numbers.
5. We give a relationship between neo balancing numbers and solutions of a certain Diophantine equation.

CHAPTER 4

RESULTS

4.1 Neo balancing numbers

Let n be a positive integer solution of

$$1+2+3+\cdots+(n-1)=(n-1)+(n+0)+(n+1)+(n+2)+\cdots+(n+r) \quad (4.1)$$

for some integer r. Then n is called the neo balancing number and r is called the neo balancer corresponding to the neo balancing number n. For example, 2, 7, 36 and 205 are neo balancing numbers with balancers -1, 1, 13 and 83, respectively. We rewrite equation (4.1) in the form

$$(n-1)^2=\frac{(n+r)(n+r+1)}{2} \quad (4.2)$$

or

$$r=\frac{-(2n+1)+\sqrt{8(n-1)^2+1}}{2}. \quad (4.3)$$

From (4.2), n is a neo balancing number if and only if $(n-1)^2$ is a triangular number. Also, by (4.3), n is a neo balancing number if and only if $8(n-1)^2+1$ is a perfect square.

The n^{th} neo balancing number is denoted by P_n, the n^{th} balancer is denoted by E_n and the n^{th} Lucas-balancing number is denoted by $Q_n=\sqrt{8(P_n-1)^2+1}$. Set initial values $B_1=2$ and $B_2=7$ and so on.

4.1.1 Function generating neo balancing numbers

In this section we will show some functions that generate next neo balancing numbers. We will show that if x is a balancing number, then

$$p(x)=3(x-1)+1+\sqrt{8(x-1)^2+1} \quad (4.4)$$

is also a neo balancing numbers.

Theorem 4.1. *For any neo balancing number x, $p(x)$ and $(p\circ p\circ\cdots\circ p)(x)$ are also* *neo balancing numbers.*

Proof. Since x is a neo balancing number, we obtain $8(x-1)^2+1$ is a perfect square and we have

$$\begin{aligned} 8(p(x)-1)^2+1 &= 8(9)(x-1)^2 + 2(8)(3)(x-1)\sqrt{8(x-1)^2+1} + 8^2(x-1)^2 + 9 \\ &= (8(x-1))^2 + 2[(8)(x-1)][(3)\sqrt{8(x-1)^2+1}] + [3\sqrt{8(x-1)^2+1}]^2 \\ &= [8(x-1) + 3\sqrt{8(x-1)^2+1}]^2 \end{aligned}$$

is a perfect square too. Then $p(x)$ is a neo balancing number. By applying $p(x)$ repeatedly, it follows that $(p \circ p \circ \cdots \circ p)(x)$ is also a neo balancing number. □

4.1.2 Finding the next neo balancing numbers

We have shown that $p(x)$ generate neo balancing numbers. In this section we will show in addition that $p(x)$ is the function generating next neo balancing numbers.

Theorem 4.2. *If x is a neo balancing number, then the next neo balancing number is*

$$p(x) = 3(x-1) + 1 + \sqrt{8(x-1)^2+1} \tag{4.5}$$

and consequently, the previous one is

$$p^{-1}(x) = 3(x-1) + 1 - \sqrt{8(x-1)^2+1}. \tag{4.6}$$

Proof. Note that $p(x)$ defines a function $p : [0,\infty) \to [2,\infty)$. Since

$$p'(x) = 3 + \frac{8(x-1)}{\sqrt{8(x-1)^2+1}} > 0,$$

p is strictly increasing. It is clear that $x < p(x)$, so p is injective. Thus p^{-1} exists. Since p is strictly increasing, p^{-1} is also strictly increasing. Let $u(x) = p^{-1}(x)$. Thus $u(x) = 3(x-1) + 1 \pm \sqrt{8(x-1)^2+1}$.

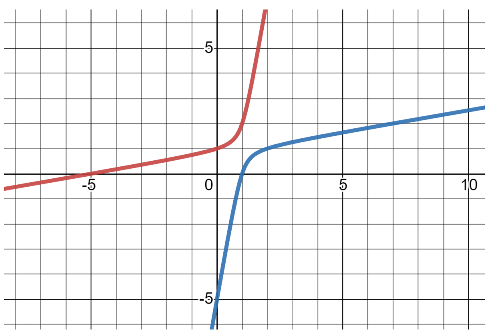

Figure 2.2 Graph of function $p(x)$ and $p^{-1}(x)$.

By figure 2.2, we have $p(x) \neq p^{-1}(x)$. Then we obtain

$$u = 3(x-1) + 1 - \sqrt{8(x-1)^2 + 1}. \tag{4.7}$$

Since n is a neo balancing number if and only if $8(n-1)^2 + 1$ is a perfect square and

$$\begin{aligned} 8(u-1)^2 + 1 &= 8(9)(x-1)^2 - 2(8)(3)(x-1)\sqrt{8(x-1)^2+1} + 8^2(x-1)^2 + 9 \\ &= (8(x-1))^2 - 2[(8)(x-1)][(3)\sqrt{8(x-1)^2+1}] + [3\sqrt{8(x-1)^2+1}]^2 \\ &= [8(x-1) - 3\sqrt{8(x-1)^2+1}]^2, \end{aligned}$$

we have $u = p^{-1}(x)$ is a neo balancing number. We let P_n be the n^{th} neo balancing number and $P_0 = 1$ such that $P_n = p(P_{n-1})$ for $n = 1, 2, \ldots$. Thus, $P_1 = 2$, $P_2 = 7$, $P_3 = 36$ and so on. Now we will prove that there is no neo balancing number between x and $p(x)$ by the method of induction.

Let H_i be the hypothesis that there is no neo balancing number between P_i and P_{i+1}. Since $P_1 = 2$ and $P_2 = 7$ it is clear that there is no neo balancing number between P_1 and P_2. Assume H_n is true. We will show that H_{n+1} is true, by contradiction. Suppose H_{n+1} is false, so there is a neo balancing number δ such that

$$P_{n+1} < \delta < P_{n+2}.$$

Thus,

$$p^{-1}(P_{n+1}) < p^{-1}(\delta) < p^{-1}(P_{n+2}).$$

We will get

$$P_n < p^{-1}(\delta) < P_{n+1}.$$

Since $p^{-1}(\delta)$ is a neo balancing number, this contradicts the induction hypothesis. So H_{n+1} is true. Therefore the neo balancing number next to x is $p(x)$. Since $p(p^{-1}(x)) = x$, it follows that $p^{-1}(x)$ is the largest neo balancing number less than x. $\square$

4.2 Properties of neo balancing numbers and balancing numbers

4.2.1 Recuurence relations between neo balancing numbers and balancing numbers

We have known that $P_0 = 1$ $P_1 = 2$, $P_2 = 7$, $P_3 = 36$, and so on. If P_n is the n^{th} neo balancing number, then

$$P_{n+1} = 3(P_n - 1) + 1 + \sqrt{8(P_n - 1)^2 + 1} \tag{4.8}$$

and

$$P_{n-1} = 3(P_n - 1) + 1 - \sqrt{8(P_n - 1)^2 + 1}. \tag{4.9}$$

It is clear from (4.8) and (4.9) that the neo balancing numbers obey the following recurrence relation:

$$P_{n+1} = 6P_n - P_{n-1} - 4 \tag{4.10}$$

or

$$\overline{P_{n+1}} = 6\overline{P_n} - \overline{P_{n-1}} \tag{4.11}$$

where $\overline{P_n} = P_n - 1$. Also, the balancing numbers obey the following recurrence relation:

$$B_{n+1} = 6B_n - B_{n-1}. \tag{4.12}$$

Using the recurrence relation (4.10), (4.11) and (4.12) we can obtain some other interesting relations concerning neo balancing numbers and balancing numbers.

Theorem 4.3. *Let B_n be the n^{th} balancing number, P_n be the n^{th} neo balancing number and $1 \leq k \leq n$ for any positive integers n and k. Then we have the following relations.*

(a) $P_{n+1}P_{n-1} = (P_n + 5)(P_n - 1)$.

(b) $P_{n+1}P_{n-1} + 9 = (P_n + 2)^2$.

(c) $P_n = B_k\overline{P_{n-k}} - B_{k-1}\overline{P_{n-k-1}} + 1$.

(d) $P_n = B_{n-1} + 1$.

(e) $P_n \cdot B_n = P_nP_{n+1} - P_n$.

(f) $P_n = \overline{P_{k+1}} \cdot \overline{P_{n-k}} - \overline{P_k} \cdot \overline{P_{n-k-1}} + 1$.

(g) $P_{2n+1} = \overline{P_{n+1}}^2 - \overline{P_n}^2 + 1$.

(h) $P_{2n} = \overline{P_n}(\overline{P_{n+1}} - \overline{P_{n-1}}) + 1$.

Proof. From (4.8) and (4.9), it follows that

$$\begin{aligned} P_{n+1}P_{n-1} &= (3(P_n-1)+1+\sqrt{8(P_n-1)^2+1})(3(P_n-1)+1-\sqrt{8(P_n-1)^2+1}) \\ &= P_n^2+4P_n-5 \\ &= (P_n+5)(P_n-1). \end{aligned}$$

This completes the proof of (a) and the proof of (b) is analogous to (a) , i.e.,

$$\begin{aligned} P_{n+1}P_{n-1}+9 &= (3(P_n-1)+1+\sqrt{8(P_n-1)^2+1})(3(P_n-1)+1-\sqrt{8(P_n-1)^2+1})+9 \\ &= P_n^2+4P_n-4 \\ &= (P_n+2)^2. \end{aligned}$$

The proof of (c) is based on mathematical induction on k. Clearly, (c) is true for $n>1$ and $k=1$. Assume that (c) is true for $k=r$, i.e., $P_n = B_r\overline{P_{n-r}} - B_{r-1}\overline{P_{n-r-1}}+1$ Thus,

$$\begin{aligned} B_{r+1}\overline{P_{n-r-1}} - B_r\overline{P_{n-r-2}}+1 &= (6B_r-B_{r-1})\overline{P_{n-r-1}} - B_r\overline{P_{n-r-2}}+1 \\ &= 6B_r\overline{P_{n-r-1}} - B_{r-1}\overline{P_{n-r-1}} - B_r\overline{P_{n-r-2}}+1 \\ &= B_r(6\overline{P_{n-r-1}} - \overline{P_{n-r-2}}) - B_{r-1}\overline{P_{n-r-1}}+1 \\ &= B_r\overline{P_{n-r}} - B_{r-1}\overline{P_{n-r-1}}+1 \\ &= P_n. \end{aligned}$$

Therefore, (c) is true for $k=r+1$. This completes the proof of (c).

The proof of (d) follows by replacing $k=n-1$ in (c). From (d), it follows that

$$\begin{aligned} P_n \cdot B_n &= P_n(P_{n+1}-1) \\ &= P_nP_{n+1} - P_n. \end{aligned}$$

This completes the proof of (e). From (c) and (d), it follows that

$$\begin{aligned} P_n &= B_k\overline{P_{n-k}} - B_{k-1}\overline{P_{n-k-1}} + 1 \\ &= (P_{k+1}-1)\overline{P_{n-k}} - (P_k - 1)\overline{P_{n-k-1}} + 1 \\ &= \overline{P_{k+1}} \cdot \overline{P_{n-k}} - \overline{P_k} \cdot \overline{P_{n-k-1}} + 1. \end{aligned}$$

This completes the proof of (f). Finally, the proof of (g) follows by replacing n with $2n+1$ and k with n in (f). Similarly, the proof of (h) follows by replacing n with $2n$ and k with n in (f). This completes the proof of Theorem 4.3. □

4.2.2 Nonrecursive form for neo balancing numbers

In this section, we shall obtain another nonrecursive form for P_n by solving the recurrence relation (4.11) as a difference equation. We rewrite the recurrence relation (4.11) in the form

$$\overline{P_{n+1}} - 6\overline{P_n} + \overline{P_{n-1}} = 0. \tag{4.13}$$

Then we have a second-order linear homogeneous difference equation whose auxiliary equation is

$$\lambda^2 - 6\lambda + 1 = 0. \tag{4.14}$$

The roots $\lambda_1 = 3+\sqrt{8}$ and $\lambda_2 = 3-\sqrt{8}$ of (4.14) are real and unequal. Thus

$$\overline{P_n} = A\lambda_1^n + B\lambda_2^n. \tag{4.15}$$

Solving for A and B, we obtain

$$A = \frac{1}{\lambda_1-\lambda_2} \quad \text{and} \quad B = -\frac{1}{\lambda_1-\lambda_2}.$$

Substituting these values into (4.15) we get

$$\overline{P_n} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1-\lambda_2}\ ;\ n = 0,1,2,3,\ldots$$

or

$$P_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1-\lambda_2} + 1\ ;\ n = 0,1,2,3,\ldots$$

where $\lambda_1 = 3+\sqrt{8}$ and $\lambda_2 = 3-\sqrt{8}$. Then we obtain the following theorem for neo balancing numbers.

Theorem 4.4. *If P_n is the n^{th} neo balancing number, then its Binet form is*

$$\overline{P_n} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}\ ;\ n = 0, 1, 2, 3, \ldots$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$.

Theorem 4.5. *If a and b are natural numbers and $a > b$, then*

$$\overline{P_{a+b}} \cdot \overline{P_{a-b}} = (\overline{P_a} - \overline{P_b})(\overline{P_a} + \overline{P_b}).$$

Proof.

$$\begin{aligned}
\overline{P_{a+b}} \cdot \overline{P_{a-b}} &= \frac{(\lambda_1^{a+b} - \lambda_2^{a+b})(\lambda_1^{a-b} - \lambda_2^{a-b})}{(\lambda_1 - \lambda_2)^2} \\
&= \frac{\lambda_1^{2a} - \lambda_1^{a-b}\lambda_2^{a+b} - \lambda_1^{a+b}\lambda_2^{a-b} + \lambda_2^{2a}}{(\lambda_1 - \lambda_2)^2} \\
&= \frac{\lambda_1^{2a} + \lambda_2^{2a} - \lambda_1^{a-b}\lambda_2^{a-b}(\lambda_1^{2b} + \lambda_2^{2b})}{(\lambda_1 - \lambda_2)^2} \\
&= \frac{\lambda_1^{2a} - 2 + \lambda_2^{2a}}{(\lambda_1 - \lambda_2)^2} - \frac{\lambda_1^{2b} - 2 + \lambda_2^{2b}}{(\lambda_1 - \lambda_2)^2} \\
&= (\frac{\lambda_1^{a} - \lambda_2^{a}}{\lambda_1 - \lambda_2})^2 - (\frac{\lambda_1^{b} - \lambda_2^{b}}{\lambda_1 - \lambda_2})^2 \\
&= \overline{P_a}^2 - \overline{P_b}^2 \\
&= (\overline{P_a} + \overline{P_b})(\overline{P_a} - \overline{P_b}).
\end{aligned}$$

□

Theorem 4.6. *If a and b are natural numbers and $a \neq b$, then*

$$\frac{\overline{P_{a+b}}}{\overline{P_{a-b}}} = \frac{(\overline{P_a} - \overline{P_b})(\overline{P_a} + \overline{P_b})}{\overline{P_{a-b}}^2}$$

Proof.

$$\begin{aligned}\frac{\overline{P_{a+b}}}{\overline{P_{a-b}}} &= \frac{\overline{P_{a+b}}}{\overline{P_{a-b}}^2}\overline{P_{a-b}} \\ &= \frac{(\overline{P_a} - \overline{P_b})(\overline{P_a} + \overline{P_b})}{\overline{P_{a-b}}^2}\end{aligned}$$

□

Theorem 4.7. *If a and b are natural numbers and $a \neq b$, then*

$$(\overline{P_{a+b}} + \overline{P_{a-b}})^n = \sum_{r=0}^{n} \left[\frac{\binom{n}{r}\left[\sum_{s=0}^{n}\binom{n}{s}\overline{P_a}^n(-\overline{P_b})^{n-s}\right]\left[\sum_{t=0}^{n}\binom{n}{t}\overline{P_a}^n(\overline{P_b})^{n-t}\right]}{\overline{P_{a-b}}^r} \right]$$

for any integers s and t.

Proof.

$$\begin{aligned}(\overline{P_{a+b}} + \overline{P_{a-b}})^n &= \sum_{r=0}^{n}\binom{n}{r}\overline{P_{a+b}}^n\,\overline{P_{a-b}}^{n-r} \\ &= \sum_{r=0}^{n}\binom{n}{r}\left[\frac{(\overline{P_a} - \overline{P_b})^n(\overline{P_a} + \overline{P_b})^n}{\overline{P_{a-b}}^r}\right] \\ &= \sum_{r=0}^{n} \left[\frac{\binom{n}{r}\left[\sum_{s=0}^{n}\binom{n}{s}\overline{P_a}^n(-\overline{P_b})^{n-s}\right]\left[\sum_{t=0}^{n}\binom{n}{t}\overline{P_a}^n(\overline{P_b})^{n-t}\right]}{\overline{P_{a-b}}^r} \right].\end{aligned}$$

□

4.2.3 Application of neo balancing numbers to a Diophatine equation

The solutions of a Diophantine equation

$$x^2 + y^2 = z^2, \; x, y, z \in \mathbb{Z}^+ \tag{4.16}$$

are quite well known as the Pythagorean triplet. We will consider the solution of (4.16) in a particular case,

$$x^2 + (x+1)^2 = y^2. \tag{4.17}$$

In this section we will find the solution of (4.17) using neo balancing numbers as follows. Let (x, y) be a solution of (4.17). Since $2y^2 - 1 = (2x+1)^2$ and

$$\frac{(2y^2-1)2y^2}{2} = y^2(2y^2-1),$$

we obtain

$$\frac{(2y^2-1)2y^2}{2}$$

is a triangular number as well as a perfect square. Thus,

$$P \;=\; \sqrt{y^2(2y^2-1)} + 1 \tag{4.18}$$

is an even neo balancing number (since y^2 and $2y^2-1$ are odd). Since $y^2 \geq 1$, it follows from (4.18) that

$$y \;=\; \frac{\sqrt{1+\sqrt{8(P-1)^2+1}}}{2}. \tag{4.19}$$

From (4.17) and (4.19), we obtain

$$x \;=\; \frac{-2+\sqrt{2\sqrt{8(P-1)^2+1}-2}}{4}$$

For example, 2 is a neo balancing number, so we obtain

$$x = \frac{-2+\sqrt{2\sqrt{8(2-1)^2+1}-2}}{4} = 0,$$

$$y = \frac{\sqrt{1+\sqrt{8(2-1)^2+1}}}{2} = 1.$$

Thus, $(0, 1)$ is a solution of

$$x^2 + (x+1)^2 = y^2, \text{ i.e.,}$$

$$0^2 + (0+1)^2 = 1^2.$$

4.3 The connection of analogous properties of neo balancing numbers

Panda (2009) introduced some fascinating properties of balancing numbers. In this section we will introduce some properties of neo balancing numbers as follows.

4.3.1 Arithmetic properties of neo balancing numbers

We know that, if x and y are complex numbers, then

$$(x+y)(x-y) = x^2 - y^2.$$

So we obtained an analogous property of neo balancing numbers as Theorem 4.5.

If n is a natural number, then

$$1 + 3 + \cdots + (2n-1) = n^2,$$

$$2 + 4 + \cdots + 2n = n(n+1),$$

$$1 + 2 + \cdots + 2n = n(2n+1).$$

In the following theorem, we obtain properties of neo balancing numbers similar to the above three identities.

Theorem 4.8. *If n is a natural number, then*

(a) $\overline{P_1} + \overline{P_3} + \overline{P_5} + \cdots + \overline{P_{2n-1}} = \overline{P_n}^2$.

(b) $\overline{P_2} + \overline{P_4} + \overline{P_6} + \cdots + \overline{P_{2n}} = \overline{P_n} \cdot \overline{P_{n+1}}$.

(c) $\overline{P_1} + \overline{P_2} + \overline{P_3} + \cdots + \overline{P_{2n}} = \overline{P_n}(\overline{P_n} + \overline{P_{n+1}})$.

Proof. From relation (h) of Theorem 4.3, we obtain

$$\overline{P_{2n+1}} = \overline{P_{n+1}}^2 - \overline{P_n}^2,$$

so (a) follows.

From relation (i) of Theorem 4.3, we replace n by $n+1$ as follows

$$\overline{P_{2n+2}} = \overline{P_{n+2}} \cdot \overline{P_{n+1}} - \overline{P_{n+1}} \cdot \overline{P_n},$$

so (b) follows.

Finally, the identity (c) directly follows from (a) and (b). □

4.3.2 De-Moivre properties of neo balancing numbers

The complex identity

$$(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$$

is known as the de-Moivre's formula. The following theorem looks like de-Moivre's formula. We define Q_n to be the n^{th} Lucas neo balancing number given by $Q_n = \sqrt{8\overline{P_n}^2 + 1}$.

Theorem 4.9. *If n and r are natural numbers, then*

$$(Q_n + \sqrt{8}\overline{P_n})^r = Q_{nr} + \sqrt{8}\overline{P_{nr}}.$$

Proof. Form Theorem 4.4, we obtain

$$Q_n^2 = 8\overline{P_n}^2 + 1 = 8\left[\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}\right] + 1 = \left[\frac{\lambda_1^n - \lambda_2^n}{2}\right]^2.$$

Therefore,

$$Q_n = \left[\frac{\lambda_1^n - \lambda_2^n}{2}\right].$$

Since

$$Q_n + \sqrt{8}\overline{P_n} = \left[\frac{\lambda_1^n - \lambda_2^n}{2}\right] + \sqrt{8}\left[\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}\right] = \lambda_1^n,$$

we obtain

$$(Q_n + \sqrt{8}\overline{P_n})^r = (\lambda_1^n)^r = \lambda_1^{nr} = Q_{nr} + \sqrt{8}\overline{P_{nr}}.$$

□

Remark. *The Fibonacci numbers satisfy a similar property*

$$\left[\frac{L_n + \sqrt{5}F_n}{2}\right]^r = \frac{L_{rn} + \sqrt{5}F_{rn}}{2} \tag{4.20}$$

Corollary 4.10. *If n and r are natural numbers, then*

$$(Q_n - \sqrt{8}\overline{P_n})^r = Q_{nr} - \sqrt{8}\overline{P_{nr}}.$$

Proof. Since

$$Q_n - \sqrt{8}\overline{P_n} = \left[\frac{\lambda_1^n - \lambda_2^n}{2}\right] - \sqrt{8}\left[\frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}\right] = \lambda_2^n,$$

the result follows easily. $\square$

4.3.3 Trigonometric properties of neo balancing numbers

The trigonometric identity

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

is quite well known.

Theorem 4.11. *If m and n are natural numbers, then*

$$\overline{P_{m+n}} = Q_m\overline{P_n} + Q_n\overline{P_m}.$$

Proof. Since

$$(Q_m + \sqrt{8}\overline{P_m})(Q_n + \sqrt{8}\overline{P_n}) = \lambda_1^m\lambda_1^n = \lambda_1^{m+n} = Q_{m+n} + \sqrt{8}\overline{P_{m+n}}$$

and

$$(Q_m + \sqrt{8}\overline{P_m})(Q_n + \sqrt{8}\overline{P_n}) = Q_mQ_n + 8\overline{P_mP_n} + \sqrt{8}(Q_n\overline{P_m} + Q_m\overline{P_n}),$$

we compare them and equate the irrational part from both sides to obtain

$$\overline{P_{m+n}} = Q_m\overline{P_n} + Q_n\overline{P_m}.$$

$\square$

Remark. *The Fibonacci numbers satisfy a similar property*

$$F_{m+n} = \frac{F_mL_n + L_mF_n}{2}. \tag{4.21}$$

Corollary 4.12. *If m and n are natural numbers, then*

$$Q_{m+n} = Q_mQ_n + 8\overline{P_m}\,\overline{P_n}.$$

Proof. This can be proven using the arguments in the proof of Theorem 4.11. $\square$

The following corollary is similar to the trigonometric identity

$$\sin 2x = 2\sin x \cos x.$$

Corollary 4.13. *If m is natural number, then*

$$\overline{P_{2m}} = 2\overline{P_m}Q_m.$$

Proof. From Theorem 4.11, we replace n by m. □

Remark. *The Fibonacci numbers satisfy a similar property*

$$F_{2n} = F_n L_n. \tag{4.22}$$

4.3.4 Properties concerning the greatest common divisor of two balancing numbers

Theorem 4.14. *If m and n are natural numbers, then*

$$\overline{P_m} \text{ divides } \overline{P_n} \text{ if and only if } m \text{ divides } n.$$

To prove the above theorem we need the following lemmas.

Lemma 4.15. *If m and n are natural numbers, then*

$$(\overline{P_n}, Q_n) = 1.$$

Proof. Since we have defined $Q_n = \sqrt{8\overline{P_n}^2 + 1}$, we obtain $Q_n^2 = 8\overline{P_n}^2 + 1$. Then

$$(\overline{P_n}^2, Q_n^2) = 1 \text{ and thus } (\overline{P_n}, Q_n) = 1.$$

□

Lemma 4.16. *If n and k are natural numbers, then*

$$\overline{P_k} \text{ divides } \overline{P_{nk}}.$$

Proof. The proof is based on mathematical induction. Clearly, this lemma is true for $n = 1$. Assuming that this lemma is true for $n = r$, we obtain $\overline{P_k}$ divides $\overline{P_{rk}}$. By Theorem 4.11 , we have

$$\begin{aligned}\overline{P_{(r+1)k}} &= \overline{P_{rk+k}} \\ &= \overline{P_{rk}}Q_k + Q_{rk}\overline{P_k}.\end{aligned}$$

Thus, we obtain that $\overline{P_k}$ divides $\overline{P_{(r+1)k}}$. $\square$

Lemma 4.17. *If n and k are natural numbers, then*

$$(\overline{P_k}, Q_{nk}) = 1.$$

Proof. Since we have shown that $(\overline{P_{nk}}, Q_{nk}) = 1$ and $\overline{P_k}$ divides $\overline{P_{nk}}$, we obtain that

$$(\overline{P_k}, Q_{nk}) = 1.$$

$\square$

Lemma 4.18. *If n and k are natural numbers and $\overline{P_k}$ divides $\overline{P_n}$, then k divides n.*

Proof. Obviously, $n \geq k$ and this lemma is true for $n = k$. By Euclid's division lemma, there exist integers q and r such that $q \geq 1$, $0 \leq r < k$ and $n = qk + r$. By Theorem 4.11, we have

$$\overline{P_n} = \overline{P_{qk+r}} = \overline{P_{qk}} Q_r + Q_{qk} \overline{P_r}.$$

Since $\overline{P_k}$ divides $\overline{P_{qk}}$ and $(\overline{P_k}, Q_{nk}) = 1$ by previous lemmas, we obtain that

$$\overline{P_k} \text{ divides } \overline{P_r}.$$

Since $r < k$, we obtain that $\overline{P_r} = 0$. Thus, we have $r = 0$ and hence $n = qk$. Therefore k divides n. $\square$

Theorem 4.14 directly follows from Lemmas 4.16 and 4.18.

Remark. *The Fibonacci numbers satisfy a similar property*

$$F_m \textit{ divides } F_n \textit{ if and only if } m \textit{ divides } n.$$

The following theorem gives a stronger result.

Theorem 4.19. *If m and n are natural numbers, then*

$$(\overline{P_m}, \overline{P_n}) = \overline{P_{(m,n)}}.$$

Proof. If $m = n$, then the proof is trivial. Assume without loss of generality that $m < n$. By Euclid's division lemma, there exist integers q_1 and r_1 such that $q_1 \geq 1$, $0 \leq r_1 < m$ and $n = q_1 m + r_1$. By Theorem 4.11, we have

$$(\overline{P_m}, \overline{P_n}) = (\overline{P_m}, \overline{P_{q_1m+r_1}}) = (\overline{P_m}, \overline{P_{q_1m}}Q_{r_1} + Q_{q_1m}\overline{P_{r_1}}).$$

Since $\overline{P_m}$ divides $\overline{P_{q_1m}}$ and $(\overline{P_m}, Q_{q_1m}) = 1$, we have

$$(\overline{P_m}, \overline{P_n}) = (\overline{P_m}, \overline{P_{r_1}}) \text{ and } (m, n) = (m, q_1m + r_1) = (m, r_1).$$

If $r_1 > 0$, then there exist integers q_2 and r_2 such that $q_2 \geq 1$, $0 \leq r_2 < r_1$ and $m = q_2r_1 + r_2$ such that

$$(\overline{P_m}, \overline{P_n}) = (\overline{P_m}, \overline{P_{r_1}}) = (\overline{P_{q_2r_1+r_2}}, \overline{P_{r_1}}) = (\overline{P_{q_2r_1}}Q_{r_2} + Q_{q_2r_1}\overline{P_{r_2}}, \overline{P_{r_1}}).$$

Since $\overline{P_{r_1}}$ divides $\overline{P_{q_2r_1}}$ and $(\overline{P_{r_1}}, Q_{q_2r_1}) = 1$, we have

$$(\overline{P_m}, \overline{P_n}) = (\overline{P_{r_2}}, \overline{P_{r_1}}) \text{ and } (m, n) = (q_2r_1 + r_2, r_1) = (r_2, r_1).$$

The process may be continued as long as exists $r_i \neq 0$. Since $r_1 > r_2 > \cdots$, it follows that $r_i \leq m - i$, so that after at most m steps some r_i will be equal to zero. If $r_{k-1} > 0$ and $r_k = 0$, then

$$(\overline{P_m}, \overline{P_n}) = (\overline{P_{r_{k-2}}}, \overline{P_{r_{k-1}}}) = (\overline{P_{q_kr_{k-1}}}, \overline{P_{r_{k-1}}}) = \overline{P_{r_{k-1}}}$$

and

$$(m, n) = (r_{k-2}, r_{k-1}) = (q_kr_{k-1}, r_{k-1}) = r_{k-1}.$$

Thus,

$$(\overline{P_m}, \overline{P_n}) = \overline{P_{r_{k-1}}} = \overline{P_{(m,n)}}.$$

□

Remark. *The Fibonacci numbers satisfy a similar property*

$$(F_m, F_n) = F_{(m,n)}.$$

CHAPTER 5

CONCLUSION AND DISCUSSION

In this chapter, we summarize the results about neo balancing numbers that we obtain in this thesis. Firstly, we find neo balancing numbers from expected equation and their definition. Secondly, we study and prove results about neo balancing numbers, their properties and applications. Finally, we find similarity between properties of neo balancing numbers and some properties of complex number, trigonometric, Fibonacci numbers and balancing numbers.

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