

NUMERICAL SOLUTIONS OF FRACTIONAL DIFFERENTIAL LOGISTIC EQUATIONS USING THE RESIDUAL POWER SERIES METHOD WITH ADOMIAN POLYNOMIALS

## PATCHAREE DUNNIMIT

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR MASTER OF SCIENCE IN MATHEMATICS

FACULTY OF SCIENCE
BURAPHA UNIVERSITY
2020

## PATCHAREE DUNNIMIT

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR MASTER OF SCIENCE IN MATHEMATICS

FACULTY OF SCIENCE
BURAPHA UNIVERSITY
2020

The Thesis of Patcharee Dunnimit has been approved by the examining committee to be partial fulfillment of the requirements for the Master of Science in Mathematics of Burapha University
Advisory Committee Examining CommitteePrincipal advisor
......................................................
$\qquad$Chair Committee
(Asst. Prof. Dr.Duangkamol Poltem)

Co-advisor $\qquad$(Asst. Prof. Dr.Duangkamol Poltem)
$\qquad$Committee
Committee
(Asst. Prof. Dr.Sineenart Srimongkol)
Dean of the Faculty of Science

## (Asst. Prof. Dr.Ekaruth Srisook)

Date.
This Thesis has been approved by Graduate School Burapha University to be partial fulfillment of the requirements for the Master of Science in Mathematics of Burapha University
Dean of Graduate School
(Assoc. Prof. Dr.Nujjaree Chaimongkol)

Date $\qquad$

This research was supported by a grant from Science Achievement Scholarship of Thailand (SAST).

## ACKNOWLEDGEMENT

In the completion of my Master Thesis, I would like to express the deepest appreciation to my advisor, Assistant Professor Dr. Duangkamol Poltem, for support of my thesis study and research, for her attentive, motivation, enthusiasm and knowledge along with guidance and teaching so that I will apply it in the future. I am also thankful my co-advisor, Assistant Professor Dr.Araya Wiwatwanich, for her recommendation and inspirational technique to complete my research work.

I appreciate Associate Professor Dr.Pattrawut Chansangiam for taking his valuable time to serve as the principal examiner and Assistant Professor Dr.Sineenart Srimongkol for useful comments and helping to correct my thesis.

Finally, I would like to thank my family members and my friends for their support and encouragement throughout the course of the thesis. Special thanks gratefully for the scholarship, Science Achievement Scholarship of Thailand (SAST), for supporting this thesis.

61910071: MAJOR: MATHEMATICS; M.Sc. (MATHEMATICS)

## KEYWORDS : FRACTIONAL DIFFERENTIAL EQUATIONS, FRACTIONAL LOGISTIC EQUATION, POPULATION GROWTH MODEL, FRACTIONAL POWER SERIES, RESIDUAL POWER SERIES METHOD

PATCHAREE DUNNIMIT : NUMERICAL SOLUTIONS OF FRACTIONAL DIFFERENTIAL LOGISTIC EQUATIONS USING THE RESIDUAL POWER SERIES METHOD WITH ADOMIAN POLYNOMIALS. ADVISORY COMMITTEE: DUANGKAMOL POLTEM, Ph.D., ARAYA WIWATWANICH, Ph.D. 59 P. 2020.

In this thesis, a combined from of the residual power series method with the Adomian polynomial is developed for analytic treatment of the fractional logistic equations and the fractional Volterra population growth model. The Caputo operator is used to define the derivative of fractional order. The convergent analysis of solution is proposed. Illustrative examples will be examined to support the proposed analysis. The fractional order solutions are compared to the integer order solutions.

## CONTENTS

Page
CONTENTS ..... G
LIST OF TABLES ..... I
LIST OF FIGURES ..... J
CHAPTER

1. INTRODUCTION ..... 1
Logistic equation ..... 1
Fractional calculus ..... 3
Fractional logistic equation ..... 6
The Volterra population growth model ..... 6
The residual power series method ..... 8
Adomian polynomials ..... 9
Research objectives ..... 10
Scope of the study ..... 10
Outline of the study ..... 11
2. LITERATURE REVIEWS ..... 12
A review of logistic differential equation ..... 12
A review of fractional logistic equation ..... 13
A review of the Volterra population growth model ..... 16
A review of the RPS method ..... 17
3. RESEARCH METHODOLOGY ..... 18
Some basic definition and theorem for RPS method ..... 18
The RPS method for fractional logistic equation ..... 19
The RPS method for the Volterra population growth model ..... 30
4. NUMERICAL RESULTS ..... 46
5. CONCLUSION AND DISCUSSION ..... 51
REFERENCES ..... 53
BIOGRAPHY ..... 59

## LIST OF TABLES

Tables Page
4.1 Error when $\alpha=1$ ..... 47
4.2 Error when $\alpha=1$. ..... 48
4.3 Error when $\alpha=1$. ..... 49
4.4 The solution of Example 4.4 when $k=6$. ..... 50

## LIST OF FIGURES

Figures Page
4.1 The approximate solution of Example 4.1 for some $0<\alpha \leq 1$ ..... 47
4.2 The approximate solution of Example 4.2 for some $0<\alpha \leq 1$ ..... 48
4.3 The approximate solution of Example 4.3 for some $0<\alpha \leq 1$ ..... 49
4.4 The approximate solution of Example 4.4 for some $0<\alpha \leq 1$ ..... 50

## CHAPTER 1

## INTRODUCTION

Mathematical modeling is the art of describing natural phenomena and various problems in the real world situations. The concept of mathematical modeling is translating various problems into tractable mathematical formulations whose theoretical and numerical analysis provides useful answers for the that problems. The models would be analyzed to describe those problems and to utilize it to represent analyze make predictions, or provide insight into real world phenomena. One of useful mathematical modelings is the logistic growth model which applied in biological and socials science. In this chapter, the logistic equation, fractional calculus which is the useful tool for solving mathematical model, the Volterra population growth model, and the residual power series method are introduced.

## Logistic equation

Mathematical models are used extensively in science such as physics, chemistry, biology, and engineering. The study of population growth is one of the specific field of science which is gaining attention since the limitation of resources on our planet. The Malthusian growth model and the logistic growth model are simple models of population growth. Like other mathematical model, growth model could be set as linear equations or algebraic equations or differential equations.

In 1798, Malthus proposed the assumption that the population growth rate is proportional to the size of the population. It is called the Malthusian growth model or exponential growth model. The differential form of the Malthusian growth model as

$$
\begin{equation*}
\frac{d N}{d t}=\rho N, t \geq 0 \tag{1.1}
\end{equation*}
$$

where $N$ is is the size of the population with respect to time $t$, and $\rho$ is the population growth rate.

Then, the solution of equation (1.1) is

$$
N(t)=N_{0} e^{\rho t}
$$

with $N_{0}$ is the size of the population at time $t=0$.
The differential equation (1.1) has an interesting explanation. The left-hand side demonstrates the rate at which the population increases or decreases with respect to time $t$. The right-hand side is equal to a positive constant multiplied by the population size. Therefore the differential equation determines that the rate at which the population increases is proportional to the population at that point in time.

In 1838, Verhulst introduced the logistic growth model which developed from Malthusian growth model. This growth model explains population dynamics in the discipline of biological and social sciences. The logistic growth model is stated in the form of nonlinear differential equation as

$$
\begin{equation*}
\frac{d N}{d t}=\rho N\left(1-\frac{N}{K}\right), t \geq 0 \tag{1.2}
\end{equation*}
$$

where $\rho>0$ represents the maximum population growth rate, $N$ is the size of the population with respect to time $t$ and $K$ is the carrying capacity.

By letting $u=N / K$, the logistic growth model of nonlinear differential equation (1.2) becomes

$$
\begin{equation*}
\frac{d u}{d t}=\rho u(1-u), t \geq 0 \tag{1.3}
\end{equation*}
$$

Equation (1.3) is said to be logistic equation or logistic differential equation.
The exact closed form solution of equation (1.3) is given by

$$
\begin{equation*}
u(t)=\frac{u_{0}}{u_{0}+\left(1-u_{0}\right) e^{-\rho t}}, t \geq 0 \tag{1.4}
\end{equation*}
$$

where $u_{0}$ is the initial value at the time $t=0$.
In recent year, the logistic equation was interested by many researchers especially for the the logistic equation of fractional order. Therefore, we introduce some basic definition and properties about the fractional derivative.

## Fractional calculus

Fractional derivative is a part of fractional calculus which has been extensively studied in recent year. The definition of fractional derivative is extended from the definition of derivative for any positive integer order. Many researchers have attempted to find a proper definition of fractional derivatives for example Riemann-Liouville, Caputo, Comformable and Hadamard. Most of them used an integral form for the fractional derivative. Two of which are the most popular ones.

Firstly, we introduce some special function which important for fractional calculus as Gamma and Beta function. After that, we introduce definition of fractional calculus.

Definition 1.1. The Gamma function is a function satisfying the following form

$$
\begin{gathered}
\Gamma(z)=\int_{0}^{\infty} e^{-t} t^{z-1} d t, \operatorname{Re}(z)>0 \\
\text { or } \quad \Gamma(z)=\frac{\Gamma(z+n)}{z(z+1)(z+2) \cdots(z+n-1)}, \operatorname{Re}(z)>-n+1 .
\end{gathered}
$$

Definition 1.2. The Gamma function is a function satisfying the following equation

$$
B(z, w)=\int_{0}^{1} t^{z-1}(1-t)^{w-1} d t, \operatorname{Re}(z)>0, \operatorname{Re}(w)>0 .
$$

For our convenience, we use $B(z, w)$ instead of Gamma functions. The following relation between Gamma and Beta function (Gradshteyn \& Ryzhik, 1963)

$$
\begin{equation*}
B(z, w)=\frac{\Gamma(z) \Gamma(w)}{\Gamma(z+w)} \tag{1.5}
\end{equation*}
$$

The equation (1.5) was used in the proof of Theorem 1.6.

Definition 1.3. (Miller \& Ross, 1993)
The Riemann-Liouville fractional derivative operator $D_{t}^{\alpha}$ of order $\alpha$ is defined in the following form

$$
D_{t}^{\alpha} u(t)= \begin{cases}\frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{d t^{n}} \int_{0}^{t}(t-\tau)^{n-\alpha-1} u(\tau) d \tau, & n-1<\alpha<n \\ u^{(n)}(t), & \alpha=n \in \mathbb{N}\end{cases}
$$

Definition 1.4. (Miller \& Ross, 1993)
The Caputo fractional derivative operator $D_{t}^{\alpha}$ of order $\alpha$ is defined in the following form

$$
D_{t}^{\alpha} u(t)= \begin{cases}\frac{1}{\Gamma(n-\alpha)} \int_{0}^{t}(t-\tau)^{n-\alpha-1} u^{(n)}(\tau) d \tau, & n-1<\alpha<n \\ u^{(n)}(t), & \alpha=n \in \mathbb{N}\end{cases}
$$

where $\Gamma(\cdot)$ is Gamma function.

Lemma 1.5. (Podlybuy, 1999)
Let $n-1<\alpha<n, n \in \mathbb{N}, \alpha \in \mathbb{R}$ and the functions $u(t)$ and $v(t)$ be such that both $D_{t}^{\alpha} u(t)$ and $D_{t}^{\alpha} v(t)$ exist. The Caputo fractional derivative is a linear operator

$$
D_{t}^{\alpha}(\lambda u(t)+\mu v(t))=\lambda D_{t}^{\alpha} u(t)+\mu D_{t}^{\alpha} v(t)
$$

where $\lambda$ and $\mu$ are constants.

The Caputo fractional derivative is for the constant function.

$$
D_{t}^{\alpha} c=0, c \text { is a constant }
$$

Theorem 1.6. (Syam, 2017)
The Caputo fractional derivative of the power function is given by

$$
D_{t}^{\alpha} t^{r}= \begin{cases}\frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} t^{r-\alpha}, & n-1<\alpha<n, r>n-1, r \in \mathbb{R} \\ 0, & n-1<\alpha<n, r \leq n-1, r \in \mathbb{N}\end{cases}
$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Proof. The proof of the second case

$$
D_{t}^{\alpha} t^{r}=0, n-1<\alpha<n, r \leq n-1, r \in \mathbb{N}
$$

follows the pattern of the proof of the differentiation of the constant function, since $\left(t^{r}\right)^{(n)}$ for $r \leq n-1, r, n \in \mathbb{N}$.

The more interesting case is the first one. It can be proved directly, using the definition of the Caputo fractional derivative (1.4) and the properties of Gamma function and Beta function (1.5).

Let $n-1<\alpha<n, r>n-1, r \in \mathbb{R}$.

$$
\begin{aligned}
D_{t}^{\alpha} t^{r} & =\frac{1}{\Gamma(n-\alpha)} \int_{0}^{t}(t-\tau)^{n-\alpha-1}\left(\tau^{r}\right)^{(n)} d \tau \\
& =\frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} \frac{\Gamma(r+1)}{\Gamma(r-n+1)}\left(\tau^{r-n}\right)(t-\tau)^{n-\alpha-1} d \tau
\end{aligned}
$$

and using the substitution $\tau=\lambda t, 0 \leq \lambda \leq 1$

$$
\begin{aligned}
D_{t}^{\alpha} t^{r} & =\frac{\Gamma(r+1)}{\Gamma(n-\alpha) \Gamma(r-n+1)} \int_{0}^{1}(\lambda t)^{r-n}((1-\lambda) t)^{n-\alpha-1} t d \lambda \\
& =\frac{\Gamma(r+1)}{\Gamma(n-\alpha) \Gamma(r-n+1)} t^{r-\alpha} \int_{0}^{1} \lambda^{r-n}(1-\lambda)^{n-\alpha-1} d \lambda \\
& =\frac{\Gamma(r+1)}{\Gamma(n-\alpha) \Gamma(r-n+1)} t^{r-\alpha} B(r-n+1, n-\alpha) \\
& =\frac{\Gamma(r+1)}{\Gamma(n-\alpha) \Gamma(r-n+1)} t^{r-\alpha} \frac{\Gamma(r-n+1) \Gamma(n-\alpha)}{\Gamma(r-\alpha+1)} \\
& =\frac{\Gamma(r+1)}{\Gamma(r-\alpha+1)} t^{r-\alpha} .
\end{aligned}
$$

Definition 1.7. Let $t \geq 0$ and $u(t)$ be a function defined on $(0, t]$. Then, the RiemannLiouville fractional integral operator of order $\alpha>0$ of a function $u$, is defined as

$$
\begin{aligned}
I^{\alpha} u(t) & =\frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{u(\tau)}{(t-\tau)^{1-\alpha}} d \tau \\
I^{0} u(t) & =u(t)
\end{aligned}
$$

where $I^{\alpha}$ denotes the Riemann-Liouville fractional integral operator of order $\alpha>0$.
Theorem 1.8. The Riemann Liouville fractional integral operator of power function is given by

$$
I^{\alpha} t^{r}=\frac{\Gamma(r+1)}{\Gamma(r+\alpha+1)} t^{r+\alpha}
$$

## Fractional logistic equation

Mathematical model of fractional calculus has been achieving great appreciation owing to its importance and applies in biology (Qureshi et al., 2019), in economy (Almeida, Malinowska, \& Monteiro, 2018), and in physic (Awadalla, \& Yameni, 2018). The fractional logistic model is one of mathematical modeling which has received the attention of many researchers. This model can be acquired by using the Caputo fractional derivative operator. The fractional logistic equation is firstly introduced by ElSayed, El-Mesiry and El-Saka (2007) which is in the form

$$
D_{t}^{\alpha} u(t)=\rho u(1-u), t>0, \rho>0
$$

with the initial condition

$$
u(0)=u_{0}, u_{0}>0 .
$$

Later, West (2015) studied the fractional logistic equation is in the form

$$
D_{t}^{\alpha} u(t)=\rho^{\alpha} u(1-u), t>0, \rho>0,
$$

with the initial condition

$$
u(0)=u_{0}, u_{0}>0 .
$$

where $D_{t}^{\alpha}$ denotes the Caputo fractional derivative operator with the fractional order $0<\alpha<1$.

## The Volterra population growth model

In addition, an important model of population model was presented by Volterra (Scudo, 1971). The model is the population growth of species within a closed system namely the Volterra's population growth model which defined as

$$
\begin{gather*}
\frac{d p}{d t}=a p-b p^{2}-c p \int_{0}^{t} p(\tau) d \tau  \tag{1.6}\\
p(0)=p_{0}
\end{gather*}
$$

where $p(t)$ is the population at time $t, p_{0}$ is the initial population, $a>0$ is the birth rate coefficient, $b>0$ is the competition between species, $c>0$ is the toxicity coefficient. This model includes the well-known term of a logistic equation if $c=0$. Furthermore, it
includes an integral term $c p \int_{0}^{t} p(\tau) d \tau$ that describes the accumulated toxicity produced since time zero. The individual mortality rate is proportional to this integral, and so the population mortality rate due to toxicity must include a factor $p$. The presence of the toxic term cause the population level to decrease to a zero in the long run due to the system being closed always.

The time and population in equation (1.6) have been scaled by introducing the non-dimensional variables

$$
t=\frac{t c}{b}, \quad u=\frac{p b}{a}
$$

which provide the non-dimensional problem

$$
\begin{gather*}
\kappa \frac{d u}{d t}=u-u^{2}-u \int_{0}^{t} u(\tau) d \tau  \tag{1.7}\\
u(0)=u_{0}
\end{gather*}
$$

where $\kappa=\frac{c}{a b}$ is a prescriptive non-dimensional parameter and $u(t)$ is the scaled population of the identical individuals at time $t$. The analytic solution of equation (1.7) (TeBeest, 1997)

$$
\begin{equation*}
u(t)=u_{0} \exp \left(\frac{1}{\kappa} \int_{0}^{t}\left[1-u(\tau)-\int_{0}^{\tau} u(s) d s\right] d \tau\right) \tag{1.8}
\end{equation*}
$$

shows that for all $t, u(t)>0$ when $u_{0}>0$.
Now, we are interest the fractional Volterra population growth model as follows

$$
\begin{equation*}
\kappa D_{t}^{\alpha} u(t)=u(t)-u^{2}(t)-u(t) I^{\alpha} u(t), \alpha \in(0,1] \tag{1.9}
\end{equation*}
$$

with the initial condition

$$
\begin{equation*}
u(0)=u_{0}, u_{0}>0, \tag{1.10}
\end{equation*}
$$

and $\kappa>0$ is the identical parameter of model (1.7) and $u(t)$ is the scaled population of identical individuals at time $t$. The derivative in the fractional Volterra population growth model (1.7) is in the Caputo sense and $I^{\alpha} u(t)$ is the Riemann-Liouville fractional integral operator of order $\alpha>0$.

As the fractional logistic equation and fractional Volterra population growth model are a non-linear equation. To make it is easier to find the solution. We apply Adomian polynomials together with the residual power series method.

## The residual power series method

In general, numerical and analytical techniques have been widely used for solving linear or nonlinear differential equations including the fractional differential equation. One interesting method to solve the fractional differential equations is the residual power series method.

Recently, Abu Arqub (2013) has introduced the method for solving linear or nonlinear differential equations that is residual power series method (RPSM). This method is basically based on the combination of the power series and residual function. Normally, the coefficient of power series was calculated by comparing the coefficients of the related to terms and a recurrence relation. However, the RPSM computes the coefficients of power series by a chain of equations of one or more variables. Consider the fractional differential equation

$$
\begin{equation*}
D^{\alpha} u(t)=R(u(t))+F(u(t)), 0<t<T, \tag{1.11}
\end{equation*}
$$

with initial condition

$$
\begin{equation*}
u(0)=u_{0}, \tag{1.12}
\end{equation*}
$$

where $R(u(t))$ is linear term, $F(u(t))$ is nonlinear term and $D_{t}^{\alpha}$ denotes the Caputo fractional derivative operator with the fractional order $0<\alpha<1$.

First, let $u(t)$ be the solution of the fractional differential equation (1.11) as a fractional power series about $t=0$ of the form

$$
\begin{equation*}
u(t)=\sum_{n=0}^{\infty} \frac{u_{n} t^{\alpha \alpha}}{\Gamma(1+n \alpha)} . \tag{1.13}
\end{equation*}
$$

After that, we approximate $u(t)$ in equation (1.13) by

$$
\begin{equation*}
u_{k}(t)=\sum_{n=0}^{k} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)}, k=1,2,3, \ldots \tag{1.14}
\end{equation*}
$$

From equation (1.14), the $k^{t h}$ residual power series approximation $u_{k}(t)$ will be obtained by computing the component $u_{1}, u_{2}, \ldots, u_{k}$. Before computing these components, we define the residual function

$$
\begin{equation*}
\operatorname{Res}_{k}(t)=D_{t}^{\alpha} u_{k}(t)-R\left(u_{k}(t)\right)-F\left(u_{k}(t)\right) . \tag{1.15}
\end{equation*}
$$

Now, we have to find the coefficients $u_{1}, u_{2}, \ldots, u_{k}$ of the RPS solution (1.14) by substituting the approximation $u_{k}(t)$ into the equation (1.15) and then we solve equation

$$
\begin{equation*}
D_{t}^{(k-1) \alpha} \operatorname{Res}_{k}(0)=0, k=1,2,3, \ldots \tag{1.16}
\end{equation*}
$$

As mentioned above, the RPS method has been used for nonlinear equation. In this thesis, the Adomian polynomials are used for the nonlinear term in an easy way. Therefore, we introduce the Adomian polynomials in the next section.

## Adomian polynomials

Adomian polynomials decompose a function $u(t)$ into a sum of components

$$
u(t)=\sum_{n=0}^{\infty} v_{n}(t) .
$$

A nonlinear operator $F$ of $u(t)$ can be written in the form of

$$
F(u(t))=\sum_{n=0}^{\infty} A_{n},
$$

where $A_{n}$ are known as the Adomian polynomials determined formally from the relation

$$
A_{n}=\frac{1}{n!}\left[\frac{d^{n}}{d \lambda^{n}}\left[F\left(\sum_{i=0}^{\infty} \lambda^{i} v_{i}\right)\right]\right]_{\lambda=0}
$$

Then, the first few polynomials are given by

$$
\begin{aligned}
& A_{0}=F\left(v_{0}\right) \\
& A_{1}=v_{1} F^{\prime}\left(v_{0}\right), \\
& A_{2}=v_{2} F^{\prime}\left(v_{0}\right)+\frac{1}{2!} v_{1}^{2} F^{\prime \prime}\left(v_{0}\right), \\
& A_{3}=v_{3} F^{\prime}\left(v_{0}\right)+v_{1} v_{2} F^{\prime \prime}\left(v_{0}\right)+\frac{1}{3!} v_{1}^{3} F^{\prime \prime \prime}\left(v_{0}\right), \\
& A_{4}=v_{4} F^{\prime}\left(v_{0}\right)+\left(\frac{1}{2!} v_{2}^{2}+v_{1} v_{3}\right) F^{\prime \prime}\left(v_{0}\right)+\frac{1}{2!} v_{1}^{2} v_{2} F^{\prime \prime \prime}\left(v_{0}\right)+\frac{1}{4!} v_{1}^{4} F^{\prime \prime \prime}\left(v_{0}\right) .
\end{aligned}
$$

Other polynomials can be calculated in similar manner (Wazwaz, 2000).

## Research objectives

This research aims to

1. study the logistic equation, fractional logistic equation and the Volterra population growth model,
2. solve the fractional logistic equation and the fractional Volterra population growth model by using the residual power series method with Adomian polynomials,
3. compare the approximate solution of fractional logistic equation obtained from the residual power series method with the exact solution when the order of derivative is one.

## Scope of the study

We solve the fractional logistic equation in the form

$$
D_{t}^{\alpha} u(t)=\rho^{\alpha} u(1-u), t>0, \rho>0,
$$

with the initial condition

$$
u(0)=u_{0},
$$

where $D_{t}^{\alpha}$ denotes the Caputo fractional derivative of order $0<\alpha \leq 1$. and solve the Volterra population growth model in the form

$$
\kappa D_{t}^{\alpha} u(t)=u(t)-u^{2}(t)-u(t) I^{\alpha} u(t), \alpha \in(0,1],
$$

with the initial condition

$$
u(0)=u_{0}, u_{0}>0,
$$

where $D_{t}^{\alpha}$ denotes the Caputo fractional derivative of order $0<\alpha \leq 1$ and $I^{\alpha} u(t)$ is the Riemann-Liouville fractional integral operator of order $\alpha>0$.

## Outline of the study

This thesis is organized as follows: A background of logistic equation, the Volterra population growth model and the residual power series method including our research objectives and scope of the study are introduced in Chapter 1. In Chapter 2, we review the methods used to solve the fractional logistic equation and the Volterra population growth model. In Chapter 3, we prove the convergence analysis and find a numerical solution. In Chapter 4, we present some numerical examples to show the efficiency of the proposed method and compare the results obtained from the exact solution when the order of derivative is one. Finally, the conclusion and discussion of this research and future research work are summarized in Chapter 5.

## CHAPTER 2

## LITERATURE REVIEWS

In this chapter, we review the application of the logistic differential equation and the method to solve the fractional logistic equation including the RPS method to solve various equation.

## A review of logistic differential equation

Some literature concerned in this section are all about the logistic equation in the various form including applying the various method to solve the logistic equation. In 1838, Verhulst presented the model which described the self-limiting growth of a biological population. This model can be described by the differential equation in the form $d u / d t=\rho(1-u)$ which is called the logistic equation or logistic differential equation. This differential equation can be solved by separation of variables. The solution is $u(t)=\frac{u_{0}}{u_{0}+\left(1-u_{0}\right) e^{-\rho t}}, t \geq 0$, where $u_{0}$ is the initial state at the time $t=0$.

A typical application of the logistic equation is a common model of population growth appear in the discipline of biological and social sciences presented by Kooi, Boer, and Kooijiman (1998). In 2003, Foryś and Marciniak-Czochra presented some approaches to tumor growth modeling using the logistic equation. The application of the tumor growth model related to the logistic equation is extensively used in the framework of ecology. In addition, the constant population growth rate which does not include the limitation on food supply or spread of diseases was described by the solution of the logistic equation (Pastijin, 2006). Not only the logistic equation appears in field biological, social science and ecology but also this equation appear in the field business and economics instance product diffusion and market acceptance, an inflation rate of goods, purchasing power of peso, and employment and unemployment in the Philippines (Ramos, 2013). In addition, the researchers have studied about logistic differential equation in many aspects. For example, Winley (2007) studied
the logistic growth model and they have inspected three aspects related to the logistic growth model: (a) properties of its graphical showing under various initial conditions; (b) the relationship of the logistic model to analogous difference equation models; and (c) the logistic differential equation is used to develop of stochastic models. Subsequently, Petropoulou (2010) proposed a discrete model equivalent to the logistic differential equation. The functional-analytic method is applied to find the discrete equivalent equation which is one of the Volterra convolution types. Later, Mir and Dubeau (2016) studied the effects of some properties of the carrying capacity on the solution of the linear and logistic differential equations. They obtained new results on the behavior and the asymptotic behavior of any solutions. In 2018, Windarto, Eridani and Purwati presented a new mathematical growth model namely a WEP-modified logistic growth model. The model was derived from a modification of the classical logistic differential equation. The WEP-modified logistic growth model described growth function of a living organism. This model could be used as an alternative model to describe poultry growth curve or individual growth.

The logistic equation is one of the equations that has received attention from many researchers. Especially, the fractional logistic equation was mentioned in the various literature. Therefore, many researchers apply various methods to solve the fractional logistic equation which will be discussed in the next subsection.

## A review of fractional logistic equation

Mathematical models of the fractional differential equations have been widely applied with some engineering and industrial problems. In addition, the mathematical models were mentioned in the field physics, chemistry, economics, biophysics, polymer rheology, aerodynamics, signal processing, blood flow phenomena, electrodynamics, control theory and many others (Hilfer, 2000; Kilbas, 2006; Miller \& Ross; Oldham \& Spanier, 1974; Podlubny, 1999). For this reason, it leads to the interest of many researchers.

One of the fractional differential equations are received attention from many researchers namely fractional logistic equation which appeared in the literature (ElSayed, El-Mesiry, \& El-Saka, 2007), and (Momani \& Qaralleh, 2007). In research of El-Sayed et al. (2007), the authors studied the stability, existence and uniqueness of the solution of fractional logistic equation including providing the solution of fractional logistic equation in the form $D_{t}^{\alpha} u(t)=\rho u(t)(1-u(t))$ where $D_{t}^{\alpha}$ is Caputo fractional differential operator with the fractional order $0<\alpha<1$. The PECE (Predict, Evaluate, Correct, Evaluate) method is used to find an approximate solution to this equation. A few years later, many researchers have applied various methods for solving fractional logistic equation. Several numerical method are used to solve fractional logistic equations.

In 2012, a new iterative method (NIM) was interested by Bhalekar and DaftardarGejji to solve the fractional logistic equation. They compared the results obtained by a new iterative method, adomian decomposition method (ADM) (Momani \& Qaralleh, 2007), and homotopy perturbation method (HAM) with exact solution. In the same year, Sweilam, Khader and Mahdy (2012) have been utilized finite difference method and variational iteration method to solve the fractional logistic equation and they have been represented the numerical results. In 2013, an approximate formula of fractional derivatives was introduced by Khader and Babatin. This formula is based on the generalized Laguerre polynomials which applied to solve the fractional logistic equations. It so-called the new spectral Laguerre collocation method. In addition, many researchers still used other methods for solving the fractional logistic equation such as Mohamed (2014) used the optimal homotopy analysis method (OHAM) used to find approximate solutions of the fractional logistic equation. And in the next year, the method was presented which for solving the fractional logistic equation namely the spectral iterative method (Shoja, Babolian \& Vahidi, 2015) and the operational matrices of Bernstein polynomials (Khan et al., 2015). In 2016, Khader applied fractional Chebyshev finite difference method to find the solution of the fractional logistic equation and they studied the convergence analysis and estimate the error approximation formula. After that, Vivek, Kanagarajan and Harikrishnan (2016) presented numerical solutions of the
fractional logistic equation by fractional Eulers method and Saad and AL-Shomrani (2016) compared the approximate analytical method consist of adomian decomposition method (ADM), variational iteration method, homotopy analysis method (HAM) and homotopy analysis transform method (HAM) with the exact solution of fractional logistic equation. Furthermore, in the recent years the researchers still interested in solving the fractional logistic equation by using fractional differential transform method (FDTM) and showed the efficacy of the results (Günerhan, 2019).

As mention above, many researchers utilized various method to solve fractional logistic equation which is composed of the fractional derivatives is in the Caputo sense. In 2019, the Hadamard derivative and integral formula are used in the logistic equation by Noupoue, Tangdogdu and Awadalla. They studied the existence and uniqueness of the solution of the fractional logistic equation. In addition, they computed the numerical solution of fractional logistic equation by mean of three numerical methods consist of the Letnikov method (LM) (Petráš, 2011), the generalized Euler method (Odibi \& Momani, 2008 ), and Caputo-Fabrizio (CF) method (Atangana \& Owolabi, 2018).

Another form of the fractional logistic equation is constructed by West (2015) which it is in form $D_{t}^{\alpha} u(t)=\rho^{\alpha} u(t)(1-u(t))$ where $D_{t}^{\alpha}$ is Caputo fractional derivative of order $0<\alpha<1$. He proposed the exact solution which is in form of a series of Mittag-Leffer functions by using the Carleman embedding technique. And the solution is called West function. Subsequently, Area, Losada, and Nieto (2016) claimed that the exact solution in the article of West (2015) or West function is valid only when the order of the derivative is one. So, the West function is not an exact solution for the fractional logistic equation. However, the West function was proven valid by D'Ovidio, Loreti, and Arabi (2018) subject to changing the structure of the fractional logistic equation for what they called modified fractional logistic equation .

Therefore our purpose is to find the solution of the fractional logistic equation is in form $D_{t}^{\alpha} u(t)=\rho^{\alpha} u(t)(1-u(t))$ where $D_{t}^{\alpha}$ is Caputo fractional derivative of order $0<\alpha<1$ which introduced by West (2015). In fact, the fractional logistic equation is non-linear equation which mostly uses numerical methods to find the solution. The Adomian polynomial is an interesting polynomial that makes solving non-linear equations easier. So, we are interested in solving the fractional logistic equation by the residual power series method with Adomian polynomials.

## A review of the Volterra population growth model

An interesting model which received attention from many researcher is the nonlinear Volterra population growth model. This model can explain the the population of a species within a closed system. The literature concerned in this section are all about the nonlinear Volterra population growth model. In 2006, Momani and Qaralleh presented efficient numerical algorithm for approximate solution of fractional population growth model in a closed system. The algorithm is based on Adomian's decomposition method and the solutions are calculated in the form of a convergent series. Then the Padé approximants are used in the analysis to capture the essential behavior of population. Many years later, Majid and Kajani (2013) applied a multi-domain LegengreGauss pseudospectral method for approximate solutions of the fractional Volterra model for population growth of a species in a closed system. This work shows that the proposed method is a very efficient and powerful tool for solving integro-differential equations of both integer and fractional orders. In 2016, Parand and Delkhosh introduced a new numerical approximation for solving the fractional Volterra's model for population growth of a species in a closed system. This method is based on the generalized fractional order Chebyshev orthogonal functions of first kind and the collocation method. The effectiveness of the implementation of the fractional Legendre functions with a pseudospectral numerical solution of a fractional population growth of species within a closed system is presented by Hicdurmaz and Can (2017).

## A review of the RPS method

The RPS method has been widely used by many researchers for solving linear and nonlinear differential equations of integer and fractional orders. For instance, in 2013, the RPS method was applied to solve the higher-order initial value problems by Abu Arqub et al. (Abu Arqub, Abo-Hammour, Al-Badarneh \& Momani, 2013). Later, Kumar, A., Kumar, S. and Singh (2016) presented the approximate solution of the Sharma-Tasso-Olever equation of fractional order by using this method. During the next year, Tariq and Akram (2017) presented the approximate solution of the nonlinear time space-fractional Benney-Lin by using the RPS method. In the same year, Syam (2017) proposed a numerical solution of fractional Lienards equation by using the RPS method. He found that the RPS method is very efficient for solving the fractional differential equations. In addition, the RPS method was applied to solve differential equations and fractional differential equations. For example, nonlinear boundary-layer equations (Shatnawi, 2016), time-fractional model of Vibration equation (Jena \& Chakraverty, 2019), time-fractional foam drainage equation (Alquran, 2015), time-fractional KdV equation (Senol \& Ata, 2018), time-fractional gas dynamics equation (Ramesh Rao, 2018), and so on.

## CHAPTER 3

## RESEARCH METHODOLOGY

In this chapter, we apply the residual power series method to solve the fractional logistic equation with adomian polynomials and prove the convergence of this method.

## Some basic definition and theorem for RPS method

Definition 3.1. (Miller \& Ross, 1993)
Let $n$ be the smallest integer greater than or equal to $\alpha$. The Caputo fractional derivative of order $\alpha>0$ is defined as

$$
D_{t}^{\alpha} u(t)= \begin{cases}\frac{1}{\Gamma(n-\alpha)} \int_{0}^{t}(t-\tau)^{n-\alpha-1} u^{(n)}(\tau) d \tau, & n-1<\alpha<n \\ u^{(n)}(t), & \alpha=n \in \mathbb{N}\end{cases}
$$

where $\Gamma(\cdot)$ is Gamma function.

Theorem 3.2. (Syam, 2017)
The Caputo fractional derivative of the power function is given by

$$
D_{t}^{\alpha} t^{p}= \begin{cases}\frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}, & n-1<\alpha<n, p>n-1, p \in \mathbb{R} \\ 0, & n-1<\alpha<n, p \leq n-1, p \in \mathbb{N}\end{cases}
$$

where $\Gamma(\cdot)$ is Gamma function.
Definition 3.3. (El-Ajou, Abu Arqub, Al zhour \& Momani, 2013)
A power series expansion of the form

$$
\sum_{m=0}^{\infty} c_{m}\left(t-t_{0}\right)^{m \alpha}=c_{0}+c_{1}\left(t-t_{0}\right)^{\alpha}+c_{2}\left(t-t_{0}\right)^{2 \alpha}+\cdots
$$

where $0 \leq n-1<\alpha \leq n, t \geq t_{0}$, is called fractional power series (FPS) about $t=t_{0}$.

Theorem 3.4. (El-Ajou et al., 2013)
Suppose that f has a fractional power series represent at $t=t_{0}$ of the form

$$
f(t)=\sum_{m=0}^{\infty} c_{m}\left(t-t_{0}\right)^{m \alpha}, 0 \leq n-1<\alpha \leq n, t_{0} \leq t<t_{0}+R
$$

where $R$ is the radius of convergence.
If $D^{m \alpha} f(t), m=0,1,2, \ldots$ are continuous on $\left(t_{0}, t_{0}+R\right)$, then $c_{m}=\frac{D^{m \alpha} f\left(t_{0}\right)}{\Gamma(1+m \alpha)}$.
Definition 3.5. Let $t \geq 0$ and $f$ be a function defined on $(0, t]$. Then, the RiemannLiouville fractional integral operator of order $\alpha>0$ of a function $u$, is defined as

$$
\begin{aligned}
I^{\alpha} u(t) & =\frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{u(\tau)}{(t-\tau)^{1-\alpha}} d \tau \\
I^{0} u(t) & =u(t)
\end{aligned}
$$

where $\Gamma(\cdot)$ is Gamma function.
Theorem 3.6. The Riemann Liouville fractional integral operator of power function is given by

$$
I^{\alpha} t^{p}=\frac{\Gamma(p+1)}{\Gamma(p+\alpha+1)} t^{p+\alpha}
$$

## The RPS method for fractional logistic equation

Consider the fractional logistic equation

$$
\begin{equation*}
D_{t}^{\alpha} u(t)=\rho^{\alpha} u(1-u), \alpha \in(0,1], \tag{3.1}
\end{equation*}
$$

with the initial condition

$$
\begin{equation*}
u(0)=u_{0}, u_{0}>0, \tag{3.2}
\end{equation*}
$$

and $\rho>0$. The derivative in fractional logistic equation (3.1) is in the Caputo sense.

## Algorithm to find the solution

According to the RPS method, let $u(t)$ be the solution of fractional logistic equation as a fractional power series about $t=0$ of the form

$$
\begin{equation*}
u(t)=\sum_{n=0}^{\infty} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)} \tag{3.3}
\end{equation*}
$$

The rest of our work is to find the coefficients of fractional power series.
By the initial condition (3.2), we approximate $u(t)$ in equation (3.3) by

$$
\begin{equation*}
u_{k}(t)=u_{0}+\sum_{n=1}^{k} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)}, k=1,2,3, \ldots . \tag{3.4}
\end{equation*}
$$

To find the values of the RPS-coefficient $u_{n}$, we solve the equation

$$
\begin{equation*}
D_{t}^{(n-1) \alpha} \operatorname{Res}_{n}(0)=0, n=1,2,3, \ldots \tag{3.5}
\end{equation*}
$$

where $\operatorname{Res}_{k}(t)$ is the $k$ th residual function and it defined by

$$
\begin{equation*}
\operatorname{Res}_{k}(t)=D_{t}^{\alpha} u_{k}(t)-\rho^{\alpha}\left(u_{k}(t)-u_{k}^{2}(t)\right) . \tag{3.6}
\end{equation*}
$$

Since the fractional logistic equation (3.1) is a nonlinear fractional differential equation in term $u^{2}(t)$, Adomian polynomials are implemented to calculate nonlinear term $u^{2}(t)$. So, the Adomian polynomials and the residual power series method are used together to solve the fractional logistic equation.

First, let

$$
\begin{equation*}
u_{k}(t)=\sum_{i=0}^{k} v_{i} \tag{3.7}
\end{equation*}
$$

where $v_{0}=u_{0}$ and

$$
\begin{equation*}
v_{i}=\frac{u_{i} t^{i \alpha}}{\Gamma(1+i \alpha)}, i=1,2,3, \ldots, k \tag{3.8}
\end{equation*}
$$

And let $F\left(u_{k}(t)\right)$ be the nonlinear operator

$$
\begin{equation*}
F\left(u_{k}(t)\right)=\sum_{n=0}^{\infty} A_{n} \tag{3.9}
\end{equation*}
$$

where $A_{n}$ are called Adomian polynomials determined formally from the relation

$$
\begin{equation*}
A_{n}=\frac{1}{n!}\left[\frac{d^{n}}{d \lambda^{n}}\left[F\left(\sum_{i=0}^{k} \lambda^{i} v_{i}\right)\right]\right]_{\lambda=0} \tag{3.10}
\end{equation*}
$$

From equation (3.7), we can rewritten the nonlinear polynomials $u_{k}^{2}(t)$ as

$$
F\left(u_{k}(t)\right)=\left(v_{0}+v_{1}+v_{2}+v_{3}+\cdots+v_{k}\right)^{2}=\sum_{n=0}^{\infty} A_{n} .
$$

Adomian polynomials for $F\left(u_{k}(t)\right)=u_{k}^{2}(t)$ are given by

$$
\begin{aligned}
& A_{0}=v_{0}^{2} \\
& A_{1}=2 v_{0} v_{1} \\
& A_{2}=2 v_{0} v_{2}+v_{1}^{2} \\
& A_{3}=2 v_{0} v_{3}+2 v_{1} v_{2} \\
& A_{4}=v_{2}^{2}+2 v_{1} v_{3}+2 v_{0} v_{4} \\
& A_{5}=2 v_{2} v_{3}+2 v_{0} v_{5}+2 v_{1} v_{4} \\
& A_{6}=2 v_{0} v_{6}+2 v_{1} v_{5}+2 v_{2} v_{4}+v_{3}^{2} \\
& A_{7}=2 v_{0} v_{7}+2 v_{2} v_{5}+2 v_{3} v_{4}+2 v_{1} v_{6} \\
& A_{8}=2 v_{2} v_{6}+2 v_{3} v_{5}+v_{4}^{2}+2 v_{0} v_{8}+2 v_{1} v_{7}
\end{aligned}
$$

Other polynomials can be calculated by equation (3.10) (Fatoorehchi \& Abolghasemi, 2011).

To find $u_{1}$, we substitute the first RPS approximate solution

$$
u_{1}(t)=u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}
$$

into equation (3.6) as follows

$$
\begin{aligned}
\operatorname{Res}_{1}(t)= & D_{t}^{\alpha} u_{1}(t)-\rho^{\alpha}\left(u_{1}(t)-u_{1}^{2}(t)\right) \\
= & D_{t}^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)-\rho^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) \\
& +\rho^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2} \\
= & u_{1}-\rho^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)+\rho^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2} .
\end{aligned}
$$

Then, we solve $\operatorname{Res}_{1}(0)=0$ to get

$$
\begin{equation*}
u_{1}=\rho^{\alpha}\left(u_{0}-u_{0}^{2}\right) . \tag{3.11}
\end{equation*}
$$

To find $u_{2}$, the second RPS approximate solution is in form

$$
\begin{equation*}
u_{2}(t)=u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)} . \tag{3.12}
\end{equation*}
$$

By using Adomian polynomials and $u_{2}^{2}(t)=F\left(u_{2}(t)\right)$, we have

$$
\begin{aligned}
F\left(u_{2}(t)\right) & =\left(v_{0}+v_{1}+v_{2}\right)^{2} \\
& =\sum_{n=0}^{\infty} A_{n} \\
& =A_{0}+A_{1}+A_{2}+A_{3}+A_{4} \\
& =v_{0}^{2}+2 v_{0} v_{1}+2 v_{0} v_{2}+v_{1}^{2}+2 v_{1} v_{2}+v_{2}^{2} .
\end{aligned}
$$

From $v_{0}=u_{0}$ and equation (3.8), we have

$$
\begin{align*}
u_{2}^{2}(t)=F\left(u_{2}(t)\right)= & u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}  \tag{3.13}\\
& +2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2}
\end{align*}
$$

Substituting equation (3.12) and equation (3.13) into equation (3.6) as follows

$$
\begin{aligned}
\operatorname{Res}_{2}(t)= & D_{t}^{\alpha} u_{2}(t)-\rho^{\alpha}\left(u_{2}(t)-u_{2}^{2}(t)\right) \\
= & D_{t}^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)-\rho^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right) \\
& +\rho^{\alpha}\left[u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}\right. \\
& \left.+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2}\right] .
\end{aligned}
$$

So,

$$
\begin{align*}
\operatorname{Res}_{2}(t)= & \left(u_{1}+u_{2} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)-\rho^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right) \\
& +\rho^{\alpha}\left[u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}\right.  \tag{3.14}\\
& \left.+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2}\right] .
\end{align*}
$$

Applying $D_{t}^{\alpha}$ on both sides of equation (3.14), we obtain

$$
\begin{aligned}
D_{t}^{\alpha} \operatorname{Res}_{2}(t)= & u_{2}-\rho^{\alpha}\left(u_{1}+u_{2} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) \\
& +\rho^{\alpha}\left[2 u_{0} u_{1}+2 u_{0} u_{2} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{1}^{2} \frac{\Gamma(1+2 \alpha) t^{\alpha}}{\Gamma^{3}(1+\alpha)}\right. \\
& \left.+2 u_{1} u_{2} \frac{\Gamma(1+3 \alpha) t^{2 \alpha}}{\Gamma(1+\alpha) \Gamma^{2}(1+2 \alpha)}+u_{2}^{2} \frac{\Gamma(1+4) t^{3} \alpha}{\Gamma^{2}(1+2 \alpha) \Gamma(1+3 \alpha)}\right] .
\end{aligned}
$$

Thus, we solve $D_{t}^{\alpha} \operatorname{Res}_{2}(0)=0$ to get

$$
D^{\alpha} \operatorname{Res}_{2}(0)=u_{2}-\rho^{\alpha}\left(u_{1}-2 u_{0} u_{1}\right)=0 .
$$

We have the coefficient $u_{2}$ as

$$
\begin{equation*}
u_{2}=\rho^{\alpha}\left(u_{1}-2 u_{0} u_{1}\right) . \tag{3.15}
\end{equation*}
$$

To find $u_{3}$, the third RPS approximate solution is in form

$$
\begin{equation*}
u_{3}(t)=u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)} \tag{3.16}
\end{equation*}
$$

By using Adomian polynomials and $u_{3}^{2}(t)=F\left(u_{3}(t)\right)$, we have

$$
\begin{aligned}
F\left(u_{3}(t)\right)= & \left(v_{0}+v_{1}+v_{2}+v_{3}\right)^{2} \\
= & \sum_{n=0}^{\infty} A_{n} \\
= & A_{0}+A_{1}+A_{2}+A_{3}+A_{4}+A_{5}+A_{6} \\
= & v_{0}^{2}+2 v_{0} v_{1}+2 v_{0} v_{2}+v_{1}^{2}++2 v_{0} v_{3}+2 v_{1} v_{2}+v_{2}^{2}+2 v_{1} v_{3} \\
& +2 v_{2} v_{3}+v_{3}^{2} .
\end{aligned}
$$

From $v_{0}=u_{0}$ and equation (3.8), we have

$$
\begin{align*}
u_{3}^{2}(t)= & F\left(u_{3}(t)\right) \\
= & u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2} \\
& +2 u_{0} u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2}  \tag{3.17}\\
& +2 u_{1} u_{3} \frac{t^{4 \alpha}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+2 u_{2} u_{3} \frac{t^{5 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}+\left(u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right)^{2} .
\end{align*}
$$

Substituting equation (3.16) and equation (3.17) into equation (3.6) as follows

$$
\begin{aligned}
\operatorname{Res}_{3}(t)= & D_{t}^{\alpha} u_{3}(t)-\rho^{\alpha}\left(u_{3}(t)-u_{3}^{2}(t)\right) \\
= & D_{t}^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right) \\
& -\rho^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right) \\
& +\rho^{\alpha}\left[u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}\right. \\
& +2 u_{0} u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2} \\
& \left.+2 u_{1} u_{3} \frac{t^{4 \alpha}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+2 u_{2} u_{3} \frac{t^{5 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}+\left(u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right)^{2}\right] .
\end{aligned}
$$

So,

$$
\begin{align*}
\operatorname{Res}_{3}(t)= & \left(u_{1}+u_{2} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{3} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right) \\
& -\rho^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right) \\
& +\rho^{\alpha}\left[u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}\right.  \tag{3.18}\\
& +2 u_{0} u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2} \\
& \left.+2 u_{1} u_{3} \frac{t^{4 \alpha}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+2 u_{2} u_{3} \frac{t^{5 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}+\left(u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right)^{2}\right] .
\end{align*}
$$

Applying $D_{t}^{2 \alpha}$ on both sides of equation (3.18), we get

$$
\begin{aligned}
D_{t}^{2 \alpha} \operatorname{Res}_{3}(t)= & u_{3}-\rho^{\alpha}\left(u_{2}+u_{3} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) \\
& +\rho^{\alpha}\left[2 u_{0} u_{2}+u_{1}^{2} \frac{\Gamma(1+2 \alpha)}{\Gamma^{2}(1+\alpha)}+2 u_{0} u_{3} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right. \\
& +2 u_{1} u_{2} \frac{\Gamma(1+3 \alpha) t^{\alpha}}{\Gamma^{2}(1+\alpha) \Gamma(1+2 \alpha)}+u_{2}^{2} \frac{\Gamma(1+4 \alpha) t^{2} \alpha}{\Gamma^{3}(1+2 \alpha)}+2 u_{1} u_{3} \frac{\Gamma(1+4 \alpha) t^{2 \alpha}}{\Gamma^{2}(1+\alpha) \Gamma(1+2 \alpha) \Gamma(1+3 \alpha)} \\
& \left.+u_{2} u_{3} \frac{\Gamma(1+5 \alpha) t^{3 \alpha}}{\Gamma(1+2 \alpha) \Gamma^{2}(1+3 \alpha)}+u_{3}^{2} \frac{\Gamma(1+6 \alpha) t^{4 \alpha}}{\Gamma^{2}(1+3 \alpha) \Gamma(1+4 \alpha)}\right] .
\end{aligned}
$$

Then, we solve $D_{t}^{2 \alpha} \operatorname{Res}_{3}(0)=0$ to get

$$
D_{t}^{2 \alpha} \operatorname{Res}_{3}(0)=u_{3}-\rho^{\alpha} u_{2}+\rho^{\alpha}\left(2 u_{0} u_{2}+u_{1}^{2} \frac{\Gamma(1+2 \alpha)}{\Gamma^{2}(1+\alpha)}\right)=0 .
$$

We have the coefficient $u_{3}$ as

$$
\begin{equation*}
u_{3}=\rho^{\alpha}\left(u_{2}-2 u_{0} u_{2}-u_{1}^{2} \frac{\Gamma(1+2 \alpha)}{\Gamma^{2}(1+\alpha)}\right) . \tag{3.19}
\end{equation*}
$$

To find $u_{4}$, the fourth RPS approximate solution is in form

$$
\begin{equation*}
u_{4}(t)=u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)} \tag{3.20}
\end{equation*}
$$

By using Adomian polynomials and $u_{4}^{2}(t)=F\left(u_{4}(t)\right)$, we have

$$
\begin{aligned}
F\left(u_{4}(t)\right)= & \left(v_{0}+v_{1}+v_{2}+v_{3}+v_{4}\right)^{2} \\
= & \sum_{n=0}^{\infty} A_{n} \\
= & A_{0}+A_{1}+A_{2}+A_{3}+A_{4}+A_{5}+A_{6}+A_{7}+A_{8} \\
= & v_{0}^{2}+2 v_{0} v_{1}+2 v_{0} v_{2}+v_{1}^{2}++2 v_{0} v_{3}+2 v_{1} v_{2}+v_{2}^{2}+2 v_{1} v_{3}+2 v_{0} v_{4} \\
& +2 v_{2} v_{3}+2 v_{1} v_{4}++2 v_{2} v_{4}+v_{3}^{2}+2 v_{3} v_{4}+v_{4}^{2}
\end{aligned}
$$

From $v_{0}=u_{0}$ and equation (3.8), we have

$$
\begin{align*}
u_{4}^{2}(t)= & F\left(u_{4}(t)\right) \\
= & u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2} \\
& +2 u_{0} u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha}+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2} \\
& +2 u_{1} u_{3} \frac{t^{4 \alpha}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+2 u_{0} u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}+2 u_{2} u_{3} \frac{t^{5 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}  \tag{3.21}\\
& +2 u_{1} u_{4} \frac{t^{5 \alpha}}{\Gamma(1+\alpha) \Gamma(1+4 \alpha)}+2 u_{2} u_{4} \frac{t^{6 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+4 \alpha)}+\left(u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right)^{2} \\
& +2 u_{3} u_{4} \frac{t^{7 \alpha}}{\Gamma(1+3 \alpha) \Gamma(1+4 \alpha)}+\left(u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right)^{2} .
\end{align*}
$$

Substituting equation (3.20) and equation (3.21) into equation (3.6) as follows

$$
\begin{aligned}
\operatorname{Res}_{4}(t)= & D_{t}^{\alpha} u_{4}(t)-\rho^{\alpha}\left(u_{4}(t)-u_{4}^{2}(t)\right) \\
= & D_{t}^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right) \\
& -\rho^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right) \\
& +\rho^{\alpha}\left[u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}\right. \\
& +2 u_{0} u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2} \\
& +2 u_{1} u_{3} \frac{t^{4 \alpha}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+2 u_{0} u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}+2 u_{2} u_{3} \frac{t^{5 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)} \\
& +2 u_{1} u_{4} \frac{t^{5 \alpha}}{\Gamma(1+\alpha) \Gamma(1+4 \alpha)}+2 u_{2} u_{4} \frac{t^{6 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+4 \alpha)}+\left(u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right)^{2} \\
& \left.+2 u_{3} u_{4} \frac{t^{7 \alpha}}{\Gamma(1+3 \alpha) \Gamma(1+4 \alpha)}+\left(u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right)^{2}\right] .
\end{aligned}
$$

So,

$$
\begin{align*}
\operatorname{Res}_{4}(t)= & u_{1}+u_{2} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{3} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{4} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)} \\
& -\rho^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right) \\
& +\rho^{\alpha}\left[u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}\right. \\
& +2 u_{0} u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2}  \tag{3.22}\\
& +2 u_{1} u_{3} \frac{t^{4 \alpha}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+2 u_{0} u_{4} \frac{t^{4 \alpha}}{\Gamma(+4 \alpha)}+2 u_{2} u_{3} \frac{t^{5 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)} \\
& +2 u_{1} u_{4} \frac{t^{5 \alpha}}{\Gamma(1+\alpha) \Gamma(1+4 \alpha)}+\left(u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right)^{2}+2 u_{3} u_{4} \frac{t^{7 \alpha}}{\Gamma(1+3 \alpha) \Gamma(1+4 \alpha)} \\
& \left.+\left(u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right)^{2}\right] .
\end{align*}
$$

Applying $D_{t}^{3 \alpha}$ on both sides of equation (3.22), we get

$$
\begin{aligned}
D_{t}^{3 \alpha} \operatorname{Res}_{4}(t)= & u_{4}-\rho^{\alpha}\left(u_{3}+u_{4} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) \\
& \rho^{\alpha}\left[2 u_{0} u_{3}+2 u_{1} u_{2} \frac{\Gamma(1+3 \alpha)}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+u_{2}^{2} \frac{\Gamma(1+4 \alpha) t^{\alpha}}{\Gamma(1+\alpha) \Gamma^{2}(1+2 \alpha)}\right. \\
& 2 u_{1} u_{3} \frac{\Gamma(1+4 \alpha) t^{\alpha}}{\Gamma^{2}(1+\alpha) \Gamma(1+3 \alpha)}+u_{0} u_{4} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{2} u_{3} \frac{\Gamma(1+5 \alpha) t^{2 \alpha}}{\Gamma^{2}(1+2 \alpha) \Gamma(1+3 \alpha)} \\
& 2 u_{1} u_{4} \frac{\Gamma(1+5 \alpha) t^{2 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha) \Gamma(1+4 \alpha)}+u_{3}^{2} \frac{\Gamma(1+6 \alpha) t^{3 \alpha}}{\Gamma^{3}(1+3 \alpha)}+2 u_{3} u_{4} \frac{\Gamma(1+7 \alpha) t^{4 \alpha}}{\Gamma^{3}(1+3 \alpha) \Gamma^{2}(1+4 \alpha)} \\
& \left.+u_{4}^{2} \frac{\Gamma(1+8 \alpha) t^{5 \alpha}}{\Gamma^{2}(1+4 \alpha) \Gamma(1+5 \alpha)}\right] .
\end{aligned}
$$

Thus, we solve $D_{t}^{3 \alpha} \operatorname{Res}_{4}(0)=0$ to get

$$
D_{t}^{3 \alpha} \operatorname{Res}_{4}(0)=u_{4}-\rho^{\alpha} u_{3}+\rho^{\alpha}\left[2 u_{0} u_{3}+2 u_{1} u_{2} \frac{\Gamma(1+3 \alpha)}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}\right]=0
$$

We have the coefficient $u_{4}$ as

$$
\begin{equation*}
u_{4}=\rho^{\alpha}\left(u_{3}-2 u_{0} u_{3}-u_{1} u_{2} \frac{\Gamma(1+3 \alpha)}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}\right) \tag{3.23}
\end{equation*}
$$

Using a similar argument, to find $u_{k}$ in equation (3.4).
The $k^{t h}$ RPS approximate solution is in form

$$
u_{k}(t)=u_{0}+\sum_{n=1}^{k} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)}
$$

Then,

$$
\begin{align*}
u_{k}^{2}(t)= & \sum_{n=0}^{k}\left(\sum_{i=0}^{k} \frac{u_{i} u_{n-i}}{\Gamma(1+i \alpha) \Gamma(1+(n-i) \alpha)}\right) t^{n \alpha}  \tag{3.24}\\
& +\sum_{n=1}^{k}\left(\sum_{i=n}^{k} \frac{u_{i} u_{k+n-i}}{\Gamma(1+i \alpha) \Gamma(1+(k+n-i) \alpha)}\right) t^{(k+n) \alpha}
\end{align*}
$$

We derive the $k$ th residual function as

$$
\begin{aligned}
\operatorname{Res}_{k}(t)= & D_{t}^{\alpha} u_{k}(t)-\rho^{\alpha}\left(u_{k}(t)-u_{k}^{2}(t)\right) \\
= & D_{t}^{\alpha}\left(u_{0}+\sum_{n=1}^{k} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)}\right)-\rho^{\alpha}\left(\sum_{n=0}^{k} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)}\right) \\
& +\rho^{\alpha}\left(\sum_{n=0}^{k}\left(\sum_{i=0}^{k} \frac{u_{i} u_{n-i}}{\Gamma(1+i \alpha) \Gamma(1+(n-i) \alpha)}\right) t^{n \alpha}\right. \\
& \left.+\sum_{n=1}^{k}\left(\sum_{i=n}^{k} \frac{u_{i} u_{k+n-i}}{\Gamma(1+i \alpha) \Gamma(1+(k+n-i) \alpha)}\right) t^{(k+n) \alpha}\right) .
\end{aligned}
$$

So,

$$
\begin{align*}
\operatorname{Res}_{k}(t)= & \sum_{n=1}^{k} \frac{u_{n} t^{(n-1) \alpha}}{\Gamma(1+(n-1) \alpha)}-\rho^{\alpha}\left(\sum_{n=0}^{k} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)}\right) \\
& +\rho^{\alpha}\left(\sum_{n=0}^{k}\left(\sum_{i=0}^{k} \frac{u_{i} u_{n-i}}{\Gamma(1+i \alpha) \Gamma(1+(n-i) \alpha)}\right) t^{n \alpha}\right.  \tag{3.25}\\
& \left.+\sum_{n=1}^{k}\left(\sum_{i=n}^{k} \frac{u_{i} u_{k+n-i}}{\Gamma(1+i \alpha) \Gamma(1+(k+n-i) \alpha)}\right) t^{(k+n) \alpha}\right) .
\end{align*}
$$

Now, we apply the operator $D_{t}^{(k-1) \alpha}$ on both sides of equation (3.25) becomes

$$
\begin{aligned}
D_{t}^{(k-1) \alpha} \operatorname{Res}_{k}(t)= & u_{k}-\rho^{\alpha}\left(u_{k-1}+\frac{u_{k} \alpha^{\alpha}}{\Gamma(1+\alpha)}\right) \\
& +\rho^{\alpha}\left(\sum_{i=0}^{k-1} \frac{u_{i} u_{k-1-i} \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha)}\right. \\
& +\sum_{i=0}^{k} \frac{u_{i} u_{k-i} \Gamma(1+k \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-i) \alpha) \Gamma(1+\alpha)} t^{\alpha} \\
& \left.+\sum_{n=1}^{k}\left(\sum_{i=n}^{k} \frac{u_{i} u_{k+n-i} \Gamma(1+(k+n) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k+n-i) \alpha) \Gamma(1+(n+1) \alpha)}\right) t^{(n+1) \alpha}\right) .
\end{aligned}
$$

Solving the equation $D^{(k-1) \alpha} \operatorname{Res}_{k}(0)=0$, we have

$$
D_{t}^{(k-1) \alpha} \operatorname{Res}_{k}(0)=u_{k}-\rho^{\alpha} u_{k-1}+\rho^{\alpha}\left(\sum_{i=0}^{k-1} \frac{u_{i} u_{k-1-i} \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha)}\right)=0 .
$$

The coefficient $u_{k}$ is expressed as follows

$$
\begin{equation*}
u_{k}=\rho^{\alpha}\left(u_{k-1}-\sum_{i=0}^{k-1} \frac{u_{i} u_{k-1-i} \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha)}\right) . \tag{3.26}
\end{equation*}
$$

## Convergence analysis

In this section, we prove the convergence of the residual power series method by using Lemma 3.7.

Lemma 3.7. (El-Ajou et al., 2013)
The classical power series (CPS) $\sum_{n=0}^{\infty} u_{n} t^{n},-\infty<t<\infty$, has a radius of convergence $R$ if and only if the fractional power series (FPS) $\sum_{n=0}^{\infty} u_{n} t^{n \alpha}, t \geq 0$, has a radius of convergence $R^{\frac{1}{\alpha}}$.

Proof. Consider the CPS $\sum_{n=0}^{\infty} u_{n} t^{n}$. If we make the change of variable $t=x^{\alpha}, x \geq 0$, then the CPS becomes $\sum_{n=0}^{\infty} u_{n} x^{n \alpha}$
This series converges for $0 \leq x<R$, that is $0 \leq x<R^{\frac{1}{\alpha}}$,
and so the FPS $\sum_{n=0}^{\infty} u_{n} t^{n \alpha}$ has radius of convergence $R^{\frac{1}{\alpha}}$.
Conversely, if we make the change of variable $t=x^{\frac{1}{\alpha}}, x \geq 0$,
then the FPS $\sum_{n=0}^{\infty} u_{n} t^{n \alpha}$ becomes $\sum_{n=0}^{\infty} u_{n} t^{n}, x \geq 0$.
In fact, this series converge for $0 \leq x^{\frac{1}{\alpha}}<R^{\frac{1}{\alpha}}$ that is for $0 \leq x<R$.
Since the two series $\sum_{n=0}^{\infty} u_{n} t^{n}, x \geq 0$ and $\sum_{n=0}^{\infty} u_{n} t^{n},-\infty<x<\infty$ have the same radius of convergence $R=\lim _{n \rightarrow \infty}\left|\frac{u_{n}}{u_{n+1}}\right|$, the radius of convergence for the CPS $\sum_{n=0}^{\infty} u_{n} t^{n}$, $-\infty<x<\infty$ is $R$, so the proof of this lemma is complete.

Theorem 3.8. The fractional power series solution offractional logistic equation (3.1):

$$
u(t)=\sum_{n=0}^{\infty} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)}
$$

where the coefficients are defined in equation (3.26) has a positive radius of convergence.

Proof. Since the coefficient $u_{k}$ is

$$
u_{k}=\rho^{\alpha}\left(u_{k-1}-\sum_{i=0}^{k-1} \frac{u_{i} u_{k-1-i} \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha)}\right)
$$

we can see that

$$
\frac{\left|u_{k}\right|}{\Gamma(1+k \alpha)}=\frac{\left|\rho^{\alpha}\left(u_{k-1}-\sum_{i=0}^{k-1} \frac{u_{i} u_{k-1-i} \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha)}\right)\right|}{\Gamma(1+k \alpha)}
$$

$$
\begin{aligned}
& \frac{\left|u_{k}\right|}{\Gamma(1+k \alpha)} \\
& \leq\left|\rho^{\alpha}\right|\left(\frac{\left.\left|u_{k-1}\right|+\left|\sum_{i=0}^{k-1} \frac{u_{i} u_{k-1-i} \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha)}\right|\right)}{\Gamma(1+k \alpha)}\right) \\
& \leq\left|\rho^{\alpha}\right|\left(\frac{\left|u_{k-1}\right|}{\Gamma(1+k \alpha)}+\sum_{i=0}^{k-1} \frac{\left|u_{i}\right|\left|u_{k-1-i}\right| \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha) \Gamma(1+k \alpha)}\right) \\
& \leq\left|\rho^{\alpha}\right|\left(\frac{\left|u_{k-1}\right|}{\Gamma(1+k \alpha)}+\max _{0 \leq i \leq k-1}^{k}\left\{\frac{\Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha) \Gamma(1+k \alpha)}\right\} \sum_{i=0}^{k-1}\left|u_{i}\right|\left|u_{k-1-i}\right|\right) \\
&=A\left|u_{k-1}\right|+B \sum_{i=0}^{k-1}\left|u_{i}\right|\left|u_{k-1-i}\right|
\end{aligned}
$$

where
$A=\frac{\left|\rho^{\alpha}\right|}{\Gamma(1+k \alpha)}, B=\max _{0 \leq i \leq k-1}^{k}\left\{\frac{\Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha) \Gamma(1+k \alpha)}\right\}\left|\rho^{\alpha}\right|$.
Let

$$
\begin{equation*}
g(t)=\sum_{k=0}^{\infty} a_{k} t^{k} \tag{3.27}
\end{equation*}
$$

where $a_{0}=\left|u_{0}\right|$ and

$$
\begin{equation*}
a_{k}=A a_{k-1}+B \sum_{i=0}^{k-1} a_{i} a_{k-1-i}, k=1,2, \ldots \tag{3.28}
\end{equation*}
$$

be the classical power series.
Thus,

$$
\begin{aligned}
\omega=g(t) & =a_{0}+t \sum_{k=0}^{\infty} a_{k+1} t^{k} \\
& =a_{0}+t\left(\sum_{k=0}^{\infty}\left(A a_{k}+B \sum_{i=0}^{k} a_{i} a_{k-i}\right) t^{k}\right) \\
& =a_{0}+t\left(A \sum_{k=0}^{\infty} a_{k} t^{k}+B \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a_{i} a_{k-i}\right) t^{k}\right) .
\end{aligned}
$$

Let

$$
\begin{equation*}
G(t, \omega)=\omega-a_{0}-t\left(A \omega+B \omega^{2}\right) . \tag{3.29}
\end{equation*}
$$

Then

$$
G_{\omega}(t, \omega)=1-t(A+2 B \omega) .
$$

Regarding at point $\left(0, a_{0}\right)$, the function $G(t, \omega)$ is 0 and the partial derivative of the function $G(t, \omega)$ with respect to $\omega$ is 1 . We can see that $G(t, \omega)$ is an analytic function, so $G(t, \omega)$ has continuous derivatives. By implicit function theorem (Rudin, 2004), there is a neighborhood of $\left(0, a_{0}\right)$ so that whenever $t$ is sufficiently close to 0 there is a unique $\omega$ so that $G(t, \omega)=0$. Then, $g(t)$ is an analytic function in the neighborhood of the point $\left(0, a_{0}\right)$ of the $(t, \omega)$-plane with a positive radius of convergence. From Lemma 3.7, the series in equation (3.3) converges.

## The RPS method for fractional Volterra population growth model

Consider the fractional population growth model

$$
\begin{equation*}
\kappa D_{t}^{\alpha} u(t)=u(t)-u^{2}(t)-u(t) I^{\alpha} u(t), \alpha \in(0,1], \tag{3.30}
\end{equation*}
$$

with the initial condition

$$
\begin{equation*}
u(0)=u_{0}, u_{0}>0, \tag{3.31}
\end{equation*}
$$

and $\kappa>0$ is a prescribed non-dimensional parameter and $u(t)$ is the scaled population of identical individuals at time $t$. The derivative in fractional population growth model (3.30) is in the Caputo sense and $I^{\alpha} u(t)$ is the Riemann-Liouville fractional integral operator of order $\alpha>0$.

## Algorithm to find the solution

According to the RPS method, let $u(t)$ be the solution of fractional population growth model as a fractional power series about $t=0$ of the form

$$
\begin{equation*}
u(t)=\sum_{n=0}^{\infty} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)} \tag{3.32}
\end{equation*}
$$

The rest of our work to find the coefficients of fractional power series.
By the initial condition (3.31), we approximate $u(t)$ in equation (3.32) by

$$
\begin{equation*}
u_{k}(t)=u_{0}+\sum_{n=1}^{k} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)}, k=1,2,3, \ldots . \tag{3.33}
\end{equation*}
$$

To find the values of the RPS-coefficient $u_{n}$, we solve the equation

$$
\begin{equation*}
D_{t}^{(n-1) \alpha} \operatorname{Res}_{n}(0)=0, n=1,2,3, \ldots, \tag{3.34}
\end{equation*}
$$

where $\operatorname{Res}_{k}(t)$ is the $k$ th residual function and it defined by

$$
\begin{equation*}
\operatorname{Res}_{k}(t)=\kappa D_{t}^{\alpha} u_{k}(t)-u_{k}(t)+u_{k}^{2}(t)+u_{k}(t) I^{\alpha} u_{k}(t) . \tag{3.35}
\end{equation*}
$$

Likewise, since the fractional population growth model (3.31) is a nonlinear fractional differential equation in term $u^{2}(t)$, Adomian polynomials are implemented to calculate nonlinear term $u^{2}(t)$. So, the Adomian polynomials and the residual power series method are used together to solve the fractional logistic equation.

First, let

$$
\begin{equation*}
u_{k}(t)=\sum_{i=0}^{k} v_{i} \tag{3.36}
\end{equation*}
$$

where $v_{0}=u_{0}$ and

$$
\begin{equation*}
v_{i}=\frac{u_{i} t^{i \alpha}}{\Gamma(1+i \alpha)}, i=1,2,3, \ldots, k \tag{3.37}
\end{equation*}
$$

And let $F\left(u_{k}(t)\right)$ be the nonlinear operator

$$
\begin{equation*}
F\left(u_{k}(t)\right)=\sum_{n=0}^{\infty} A_{n} \tag{3.38}
\end{equation*}
$$

where $A_{n}$ are called Adomian polynomials determined formally from the relation

$$
\begin{equation*}
A_{n}=\frac{1}{n!}\left[\frac{d^{n}}{d \lambda^{n}}\left[F\left(\sum_{i=0}^{k} \lambda^{i} v_{i}\right)\right]\right]_{\lambda=0} \tag{3.39}
\end{equation*}
$$

From equation (3.36), we can rewritten the nonlinear polynomials $u_{k}^{2}(t)$ as

$$
F\left(u_{k}(t)\right)=\left(v_{0}+v_{1}+v_{2}+v_{3}+\cdots+v_{k}\right)^{2}=\sum_{n=0}^{\infty} A_{n} .
$$

Adomian polynomials for $F\left(u_{k}(t)\right)=u_{k}^{2}(t)$ given by

$$
\begin{aligned}
& A_{0}=v_{0}^{2} \\
& A_{1}=2 v_{0} v_{1} \\
& A_{2}=2 v_{0} v_{2}+v_{1}^{2} \\
& A_{3}=2 v_{0} v_{3}+2 v_{1} v_{2} \\
& A_{4}=v_{2}^{2}+2 v_{1} v_{3}+2 v_{0} v_{4} \\
& A_{5}=2 v_{2} v_{3}+2 v_{0} v_{5}+2 v_{1} v_{4} \\
& A_{6}=2 v_{0} v_{6}+2 v_{1} v_{5}+2 v_{2} v_{4}+v_{3}^{2} \\
& A_{7}=2 v_{0} v_{7}+2 v_{2} v_{5}+2 v_{3} v_{4}+2 v_{1} v_{6} \\
& A_{8}=2 v_{2} v_{6}+2 v_{3} v_{5}+v_{4}^{2}+2 v_{0} v_{8}+2 v_{1} v_{7} .
\end{aligned}
$$

Other polynomials can be calculated by equation (3.39) (Fatoorehchi \& Abolghasemi, 2011).

To find $u_{1}$, we substitute the first RPS approximate solution

$$
u_{1}(t)=u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}
$$

into equation (3.35) as follows

$$
\begin{aligned}
\operatorname{Res}_{1}(t)= & \kappa D_{t}^{\alpha} u_{1}(t)-u_{1}(t)+u_{1}^{2}(t)+u_{1}(t) I^{\alpha}\left(u_{1}(t)\right) \\
= & \kappa D_{t}^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)-\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) \\
& +\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}+\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) I^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) \\
= & \kappa u_{1}-\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)+\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2} \\
& +\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)\left(u_{0} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{1} \frac{t^{\alpha}}{\Gamma(1+2 \alpha)}\right) .
\end{aligned}
$$

Then, we solve $\operatorname{Res}_{1}(0)=0$ to get

$$
\begin{equation*}
u_{1}=\frac{1}{\kappa}\left[u_{0}-u_{0}^{2}\right] . \tag{3.40}
\end{equation*}
$$

To find $u_{2}$, the second RPS approximate solution is in form

$$
\begin{equation*}
u_{2}(t)=u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)} . \tag{3.41}
\end{equation*}
$$

By using Adomian polynomials and $u_{2}^{2}(t)=F\left(u_{2}(t)\right)$, we have

$$
\begin{aligned}
F\left(u_{2}(t)\right) & =\left(v_{0}+v_{1}+v_{2}\right)^{2} \\
& =\sum_{n=0}^{\infty} A_{n} \\
& =A_{0}+A_{1}+A_{2}+A_{3}+A_{4} \\
& =v_{0}^{2}+2 v_{0} v_{1}+2 v_{0} v_{2}+v_{1}^{2}+2 v_{1} v_{2}+v_{2}^{2} .
\end{aligned}
$$

From $v_{0}=u_{0}$ and equation (3.37), we have

$$
\begin{align*}
u_{2}^{2}(t)=F\left(u_{2}(t)\right)= & u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}  \tag{3.42}\\
& +2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2}
\end{align*}
$$

Substituting equation (3.41) and equation (3.42) into equation (3.35) as follows

$$
\begin{aligned}
\operatorname{Res}_{2}(t)= & \kappa D_{t}^{\alpha} u_{2}(t)-u_{2}(t)+u_{2}^{2}(t)+u_{2}(t) I^{\alpha}\left(u_{2}(t)\right) \\
= & \kappa D_{t}^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)-\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right) \\
& +\left[u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}\right. \\
& \left.+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2}\right] \\
& +\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right) I^{\alpha}\left(u_{0}+u_{1 \frac{t^{\alpha}}{\Gamma(1+\alpha)}}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right) \\
= & \kappa D_{t}^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)-\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right) \\
& +\left[u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}\right. \\
& \left.+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2}\right] \\
& +\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)\left[u_{0} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{1} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{2} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right]
\end{aligned}
$$

So,

$$
\begin{align*}
\operatorname{Res}_{2}(t)= & \kappa\left(u_{1}+u_{2} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)-\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right) \\
& +\left[u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}\right. \\
& \left.+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2}\right]  \tag{3.43}\\
& +\left[\left(\frac{u_{0}^{2}}{\Gamma(1+\alpha)}\right) t^{\alpha}+\left(\frac{u_{0} u_{1}}{\Gamma^{2}(1+\alpha)}+\frac{u_{0} u_{1}}{\Gamma(1+2 \alpha)}\right) t^{2 \alpha}\right. \\
& +\left(\frac{u_{0} u_{2}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\frac{u_{1}^{2}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\frac{u_{0} u_{2}}{\Gamma(1+3 \alpha)}\right) t^{3 \alpha} \\
& \left.+\left(\frac{u_{1} u_{2}}{\Gamma^{2}(1+2 \alpha)}+\frac{u_{2} u_{2}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}\right) t^{4 \alpha}+\left(\frac{u_{2}^{2}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}\right) t^{5 \alpha}\right] .
\end{align*}
$$

Applying $D_{t}^{\alpha}$ on both sides of equation (3.43), we obtain

$$
\begin{aligned}
D_{t}^{\alpha} \operatorname{Res}_{2}(t)= & \kappa u_{2}-\left(u_{1}+u_{2} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) \\
& +\left[2 u_{0} u_{1}+2 u_{0} u_{2} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{1}^{2} \frac{\Gamma(1+2 \alpha) t^{\alpha}}{\Gamma^{3}(1+\alpha)}\right. \\
& \left.+2 u_{1} u_{2} \frac{\Gamma(1+3 \alpha) t^{2 \alpha}}{\Gamma(1+\alpha) \Gamma^{2}(1+2 \alpha)}+u_{2}^{2} \frac{\Gamma(1+4) t^{3 \alpha}}{\Gamma^{2}(1+2 \alpha) \Gamma(1+3 \alpha)}\right] \\
& +\left[u_{0}^{2}+\left(\frac{u_{0} u_{1}}{\Gamma^{2}(1+\alpha)}+\frac{u_{0} u_{1}}{\Gamma(1+2 \alpha)}\right) \frac{\Gamma(1+2 \alpha) t^{\alpha}}{\Gamma(1+\alpha)}\right. \\
& +\left(\frac{u_{0} u_{2}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\frac{u_{1}^{2}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\frac{u_{0} u_{2}}{\Gamma(1+3 \alpha)}\right) \frac{\Gamma(1+3 \alpha) t^{2 \alpha}}{\Gamma(1+2 \alpha)} \\
& \left.+\left(\frac{u_{1} u_{2}}{\Gamma^{2}(1+2 \alpha)}+\frac{u_{1} u_{2}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}\right) \frac{\Gamma(1+4 \alpha) t^{\alpha}}{\Gamma(1+3 \alpha)}+\left(\frac{u_{2}^{2}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}\right) \frac{\Gamma(1+5 \alpha) t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right] .
\end{aligned}
$$

Thus, we solve $D_{t}^{\alpha} \operatorname{Res}_{2}(0)=0$ to get

$$
D_{t}^{\alpha} \operatorname{Res}_{2}(0)=\kappa u_{2}-u_{1}+2 u_{0} u_{1}+u_{0}^{2}=0
$$

We have the coefficient $u_{2}$ as

$$
\begin{equation*}
u_{2}=\frac{1}{\kappa}\left[u_{1}-2 u_{0} u_{1}-u_{0}^{2}\right] . \tag{3.44}
\end{equation*}
$$

To find $u_{3}$, the third RPS approximate solution is in form

$$
\begin{equation*}
u_{3}(t)=u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)} \tag{3.45}
\end{equation*}
$$

By using Adomian polynomials and $u_{3}^{2}(t)=F\left(u_{3}(t)\right)$, we have

$$
\begin{aligned}
F\left(u_{3}(t)\right)= & \left(v_{0}+v_{1}+v_{2}+v_{3}\right)^{2} \\
= & \sum_{n=0}^{\infty} A_{n} \\
= & A_{0}+A_{1}+A_{2}+A_{3}+A_{4}+A_{5}+A_{6} \\
= & v_{0}^{2}+2 v_{0} v_{1}+2 v_{0} v_{2}+v_{1}^{2}++2 v_{0} v_{3}+2 v_{1} v_{2}+v_{2}^{2}+2 v_{1} v_{3} \\
& +2 v_{2} v_{3}+v_{3}^{2} .
\end{aligned}
$$

From $v_{0}=u_{0}$ and equation (3.37), we have

$$
\begin{align*}
u_{3}^{2}(t)= & F\left(u_{3}(t)\right) \\
= & u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2} \\
& +2 u_{0} u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2}  \tag{3.46}\\
& +2 u_{1} u_{3} \frac{t^{4 \alpha}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+2 u_{2} u_{3} \frac{t^{5 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}+\left(u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right)^{2} .
\end{align*}
$$

Substituting equation (3.45) and equation (3.46) into equation (3.35) as follows

$$
\begin{aligned}
\operatorname{Res}_{3}(t)= & \kappa D_{t}^{\alpha} u_{3}(t)-u_{3}(t)+u_{3}^{2}(t)+u_{3}(t) I^{\alpha}\left(u_{3}(t)\right) \\
= & \kappa D_{t}^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right) \\
& -\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right) \\
& +\left[u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}\right. \\
& +2 u_{0} u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2} \\
& \left.+2 u_{1} u_{3} \frac{t^{4 \alpha}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+2 u_{2} u_{3} \frac{t^{5 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}+\left(u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right)^{2}\right] \\
& +\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right) \\
& \times I^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right) \\
= & \kappa D_{t}^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right) \\
& -\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma \Gamma(1+3 \alpha)}\right) \\
& +\left[u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}\right. \\
& +2 u_{0} u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2} \\
& \left.+2 u_{1} u_{3} \frac{t^{4 \alpha}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+2 u_{2} u_{3} \frac{t^{5 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}+\left(u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right)^{2}\right] \\
& +\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right) \\
& \times\left[u_{0} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{1} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{2} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+u_{3} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right] .
\end{aligned}
$$

So,

$$
\begin{align*}
\operatorname{Res}_{3}(t)= & \kappa\left(u_{1}+u_{2} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{3} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right) \\
& -\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right) \\
& +\left[u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}\right. \\
& +2 u_{0} u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2} \\
& \left.+2 u_{1} u_{3} \frac{t^{4 \alpha}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+2 u_{2} u_{3} \frac{t^{5 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}+\left(u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right)^{2}\right]  \tag{3.47}\\
& +\left[\left(\frac{u_{0}^{2}}{\Gamma(1+\alpha)}\right) t^{\alpha}+\left(\frac{u_{0} u_{1}}{\Gamma^{2}(1+\alpha)}+\frac{u_{0} u_{1}}{\Gamma(1+2 \alpha)}\right) t^{2 \alpha}\right. \\
& +\left(\frac{u_{0} u_{2}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\frac{u_{1}^{2}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\frac{u_{0} u_{2}}{\Gamma(1+3 \alpha)}\right) t^{3 \alpha} \\
& +\left(\frac{u_{0} u_{3}}{\Gamma(1+3 \alpha \Gamma(1+3 \alpha)}+\frac{u_{1} u_{2}}{\Gamma^{2}(1+2 \alpha)}+\frac{u_{1} u_{2}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+\frac{u_{0} u_{3}}{\Gamma(1+4 \alpha)}\right) t^{4 \alpha} \\
& +\left(\frac{u_{1} u_{3}}{\Gamma(1+\alpha) \Gamma(1+4 \alpha)}+\frac{u_{2}^{2}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}+\frac{u_{1} u_{3}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}\right) t^{5 \alpha} \\
& \left.+\left(\frac{u_{2} u_{3}}{\Gamma(1+2 \alpha) \Gamma(1+4 \alpha)}+\frac{u_{2} u_{3}}{\Gamma^{2}(1+3 \alpha)}\right) t^{6 \alpha}+\left(\frac{u_{3}^{2}}{\Gamma(1+3 \alpha) \Gamma(1+4 \alpha)}\right) t^{7 \alpha}\right] .
\end{align*}
$$

Applying $D_{t}^{2 \alpha}$ on both sides of equation (3.47), we get

$$
\begin{aligned}
D_{t}^{2 \alpha} \operatorname{Res}_{3}(t)= & \kappa u_{3}-\left(u_{2}+u_{3} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) \\
& +\left[2 u_{0} u_{2}+u_{1}^{2} \frac{\Gamma(1+2 \alpha)}{\Gamma^{2}(1+\alpha)}+2 u_{0} u_{3} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right. \\
& +2 u_{1} u_{2} \frac{\Gamma(1+3 \alpha) t^{\alpha}}{\Gamma^{2}(1+\alpha) \Gamma(1+2 \alpha)}+u_{2}^{2} \frac{\Gamma(1+4 \alpha) t^{2 \alpha}}{\Gamma^{3}(1+2 \alpha)}+2 u_{1} u_{3} \frac{\Gamma(1+4 \alpha) t^{2 \alpha}}{\Gamma^{2}(1+\alpha) \Gamma(1+2 \alpha) \Gamma(1+3 \alpha)} \\
& \left.+u_{2} u_{3} \frac{\Gamma(1+5 \alpha))^{3 \alpha}}{\Gamma(1+2 \alpha) \Gamma^{2}(1+3 \alpha)}+u_{3}^{2} \frac{\Gamma(1+6 \alpha) t^{4 \alpha}}{\Gamma^{2}(1+3 \alpha) \Gamma(1+4 \alpha)}\right] \\
& +\left[\left(\frac{u_{0} u_{1}}{\Gamma^{2}(1+\alpha)}+\frac{u_{0} u_{1}}{\Gamma(1+2 \alpha)}\right) \frac{\Gamma(1+2 \alpha)}{\Gamma(1)}\right. \\
& +\left(\frac{u_{0} u_{2}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\frac{u_{1}^{2}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\frac{u_{0} u_{2}}{\Gamma(1+3 \alpha)}\right) \frac{\Gamma(1+3 \alpha)}{\Gamma(1+\alpha)} t^{\alpha} \\
& +\left(\frac{u_{0} u_{3}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+\frac{u_{1} u_{2}}{\Gamma^{2}(1+2 \alpha)}+\frac{u_{1} u_{2}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+\frac{u_{0} u_{3}}{\Gamma(1+4 \alpha)}\right) \frac{\Gamma(1+4 \alpha)}{\Gamma(1+2 \alpha)} t^{2 \alpha} \\
& +\left(\frac{u_{1} u_{3}}{\Gamma(1+\alpha) \Gamma(1+4 \alpha)}+\frac{u_{2}^{2}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}+\frac{u_{1} u_{3}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}\right) \frac{\Gamma(1+5 \alpha)}{\Gamma(1+3 \alpha)} t^{3 \alpha} \\
& \left.+\left(\frac{u_{2} u_{3}}{\Gamma(1+2 \alpha) \Gamma(1+4 \alpha)}+\frac{u_{2} u_{3}}{\Gamma^{2}(1+3 \alpha)}\right) \frac{\Gamma(1+6 \alpha)}{\Gamma(1+4 \alpha)} t^{4 \alpha}+\left(\frac{u_{3}^{2}}{\Gamma(1+3 \alpha) \Gamma(1+4 \alpha)}\right) \frac{\Gamma(1+7 \alpha)}{\Gamma(1+5 \alpha)} t^{5 \alpha}\right] .
\end{aligned}
$$

Then, we solve $D_{t}^{2 \alpha} \operatorname{Res}_{3}(0)=0$ to get
$D_{t}^{2 \alpha} \operatorname{Res}_{3}(0)=\kappa u_{3}-u_{2}+\left(2 u_{0} u_{2}+u_{1}^{2} \frac{\Gamma(1+2 \alpha)}{\Gamma^{2}(1+\alpha)}\right)+\left(\frac{u_{0} u_{1} \Gamma(1+2 \alpha)}{\Gamma^{2}(1+\alpha)}+u_{0} u_{1}\right)=0$.
We have the coefficient $u_{3}$ as

$$
\begin{equation*}
u_{3}=\frac{1}{\kappa}\left[u_{2}-\left(2 u_{0} u_{2}+u_{1}^{2} \frac{\Gamma(1+2 \alpha)}{\Gamma^{2}(1+\alpha)}\right)-\left(\frac{u_{0} u_{1} \Gamma(1+2 \alpha)}{\Gamma^{2}(1+\alpha)}+u_{0} u_{1}\right)\right] . \tag{3.48}
\end{equation*}
$$

To find $u_{4}$, the fourth RPS approximate solution is in form

$$
\begin{equation*}
u_{4}(t)=u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)} \tag{3.49}
\end{equation*}
$$

By using Adomian polynomials and $u_{4}^{2}(t)=F\left(u_{4}(t)\right)$, we have

$$
\begin{aligned}
F\left(u_{4}(t)\right)= & \left(v_{0}+v_{1}+v_{2}+v_{3}+v_{4}\right)^{2} \\
= & \sum_{n=0}^{\infty} A_{n} \\
= & A_{0}+A_{1}+A_{2}+A_{3}+A_{4}+A_{5}+A_{6}+A_{7}+A_{8} \\
= & v_{0}^{2}+2 v_{0} v_{1}+2 v_{0} v_{2}+v_{1}^{2}++2 v_{0} v_{3}+2 v_{1} v_{2}+v_{2}^{2}+2 v_{1} v_{3}+2 v_{0} v_{4} \\
& +2 v_{2} v_{3}+2 v_{1} v_{4}++2 v_{2} v_{4}+v_{3}^{2}+2 v_{3} v_{4}+v_{4}^{2} .
\end{aligned}
$$

From $v_{0}=u_{0}$ and equation (3.37), we have

$$
\begin{align*}
u_{4}^{2}(t)= & F\left(u_{4}(t)\right) \\
= & u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma \Gamma(1+\alpha)}\right)^{2} \\
& +2 u_{0} u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2} \\
& +2 u_{1} u_{3} \frac{t^{4 \alpha}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+2 u_{0} u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}+2 u_{2} u_{3} \frac{t^{5 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}  \tag{3.50}\\
& +2 u_{1} u_{4} \frac{t^{5 \alpha}}{\Gamma(1+\alpha) \Gamma(1+4 \alpha)}+2 u_{2} u_{4} \frac{t^{6 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+4 \alpha)}+\left(u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right)^{2} \\
& +2 u_{3} u_{4} \frac{t^{7 \alpha}}{\Gamma(1+3 \alpha) \Gamma(1+4 \alpha)}+\left(u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right)^{2} .
\end{align*}
$$

Substituting equation (3.49) and equation (3.50) into equation (3.35) as follows

$$
\begin{aligned}
\operatorname{Res}_{4}(t)= & \kappa D_{t}^{\alpha} u_{4}(t)-u_{4}(t)+u_{4}^{2}(t)+u_{4}(t) I^{\alpha}\left(u_{4}(t)\right) \\
= & \kappa D_{t}^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right) \\
& -\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right) \\
& +\left[u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}\right. \\
& +2 u_{0} u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2} \\
& +2 u_{1} u_{3} \frac{t^{4 \alpha}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+2 u_{0} u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}+2 u_{2} u_{3} \frac{t^{5 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)} \\
& +2 u_{1} u_{4} \frac{t^{5 \alpha}}{\Gamma(1+\alpha) \Gamma(1+4 \alpha)}+2 u_{2} u_{4} \frac{t^{6 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+4 \alpha)}+\left(u_{3} \frac{t^{\alpha \alpha}}{\Gamma(1+3 \alpha)}\right)^{2} \\
& \left.+2 u_{3} u_{4} \frac{t^{7 \alpha}}{\Gamma(1+3 \alpha) \Gamma(1+4 \alpha)}+\left(u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right)^{2}\right] \\
& +\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right) \\
= & \kappa I^{\alpha}\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right) \\
& -\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right) \\
& +\left[u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right) \\
& +2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2} \\
& +2 u_{0} u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2} \\
& +2 u_{1} u_{3} \frac{t^{4 \alpha}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+2 u_{0} u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}+2 u_{2} u_{3} \frac{t^{5 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)} \\
& +2 u_{1} u_{4} \frac{t^{5 \alpha}}{\Gamma(1+\alpha) \Gamma(1+4 \alpha)}+2 u_{2} u_{4} \frac{t^{6 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+4 \alpha)}+\left(u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right)^{2} \\
& \left.+2 u_{3} u_{4} \frac{t^{7 \alpha}}{\Gamma(1+3 \alpha) \Gamma(1+4 \alpha)}+\left(u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right)^{2}\right] \\
& +\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right) \\
& \times\left[u_{0} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{1} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{2} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+u_{3} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}+u_{4} \frac{t^{5 \alpha}}{\Gamma(1+5 \alpha)}\right] .
\end{aligned}
$$

So,

$$
\begin{align*}
& \operatorname{Res}_{4}(t)=\kappa\left(u_{1}+u_{2} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{3} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{4} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right) \\
& -\left(u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right) \\
& +\left[u_{0}^{2}+2 u_{0} u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{0} u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}+\left(u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)^{2}\right. \\
& +2 u_{0} u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}+2 u_{1} u_{2} \frac{t^{3 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\left(u_{2} \frac{t^{2 \alpha}}{\Gamma(1+2 \alpha)}\right)^{2} \\
& +2 u_{1} u_{3} \frac{t^{4 \alpha}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+2 u_{0} u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}+2 u_{2} u_{3} \frac{t^{5 \alpha}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)} \\
& +2 u_{1} u_{4} \frac{t^{5 \alpha}}{\Gamma(1+\alpha) \Gamma(1+4 \alpha)}+\left(u_{3} \frac{t^{3 \alpha}}{\Gamma(1+3 \alpha)}\right)^{2}+2 u_{3} u_{4} \frac{t^{7}}{\Gamma(1+3 \alpha) \Gamma(1+4 \alpha)} \\
& \left.+\left(u_{4} \frac{t^{4 \alpha}}{\Gamma(1+4 \alpha)}\right)^{2}\right]+\left[\left(\frac{u_{0}^{2}}{\Gamma(1+\alpha)}\right) t^{\alpha}+\left(\frac{u_{0} u_{1}}{\Gamma(1+2 \alpha)}+\frac{u_{0} u_{1}}{\Gamma^{2}(1+\alpha)}\right) t^{2 \alpha}\right.  \tag{3.51}\\
& +\left(\frac{u_{0} u_{2}}{\Gamma(1+3 \alpha)}+\frac{u_{1}^{2}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\frac{u_{0} u_{2}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}\right) t^{3 \alpha} \\
& +\left(\frac{u_{0} u_{3}}{\Gamma(1+4 \alpha)}+\frac{u_{1} u_{2}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+\frac{u_{1} u_{2}}{\Gamma^{2}(1+2 \alpha)}+\frac{u_{0} u_{3}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}\right) t^{4 \alpha} \\
& +\left(\frac{u_{0} u_{4}}{\Gamma(1+5 \alpha)}+\frac{u_{1} u_{3}}{\Gamma(1+\alpha) \Gamma(1+4 \alpha)}+\frac{u_{2}^{2}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}\right. \\
& \left.+\frac{u_{1} u_{3}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}+\frac{u_{0} u_{4}}{\Gamma(1+\alpha) \Gamma(1+4 \alpha)}\right) t^{5 \alpha} \\
& +\left(\frac{u_{1} u_{4}}{\Gamma(1+2 \alpha) \Gamma(1+4 \alpha)}+\frac{u_{2} u_{3}}{\Gamma^{2}(1+3 \alpha)}+\frac{u_{2} u_{3}}{\Gamma(1+2 \alpha) \Gamma(1+4 \alpha)}+\frac{u_{1} u_{4}}{\Gamma(1+\alpha) \Gamma(1+5 \alpha)}\right) t^{6 \alpha} \\
& +\left(\frac{u_{2} u_{4}}{\Gamma(1+3 \alpha) \Gamma(1+4 \alpha)}+\frac{u_{3}^{2}}{\Gamma(1+3 \alpha) \Gamma(1+4 \alpha)}+\frac{u_{2} u_{4}}{\Gamma(1+2 \alpha) \Gamma(1+5 \alpha)}\right) t^{7 \alpha} \\
& \left.+\left(\frac{u_{3} u_{4}}{\Gamma^{2}(1+4 \alpha)}+\frac{u_{3} u_{4}}{\Gamma(1+3 \alpha) \Gamma(1+5 \alpha)}\right) t^{8 \alpha}+\left(\frac{u_{4}^{2}}{\Gamma(1+4 \alpha) \Gamma(1+5 \alpha)}\right) t^{9 \alpha}\right] .
\end{align*}
$$

Applying $D_{t}^{3 \alpha}$ on both sides of equation (3.51), we get

$$
\begin{aligned}
D_{t}^{3 \alpha} \operatorname{Res}_{4}(t)= & \kappa u_{4}-\left(u_{3}+u_{4} \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right) \\
& +\left[2 u_{0} u_{3}+2 u_{1} u_{2} \frac{\Gamma(1+3 \alpha)}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+u_{2}^{2} \frac{\Gamma(1+4 \alpha) t^{\alpha}}{\Gamma(1+\alpha) \Gamma^{2}(1+2 \alpha)}\right. \\
& 2 u_{1} u_{3} \frac{\Gamma(1+4 \alpha) t^{\alpha}}{\Gamma^{2}(1+\alpha) \Gamma(1+3 \alpha)}+u_{0} u_{4} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+2 u_{2} u_{3} \frac{\Gamma(1+5 \alpha) t^{2 \alpha}}{\Gamma^{2}(1+2 \alpha) \Gamma(1+3 \alpha)} \\
& 2 u_{1} u_{4} \frac{\Gamma(1+5 \alpha) t^{2 \alpha}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha) \Gamma(1+4 \alpha)}+u_{3}^{2} \frac{\Gamma(1+6 \alpha) t^{3 \alpha}}{\Gamma^{3}(1+3 \alpha)}+2 u_{3} u_{4} \frac{\Gamma(1+7 \alpha) t^{3 \alpha}}{\Gamma^{3}(1+3 \alpha) \Gamma^{2}(1+4 \alpha)} \\
& \left.+u_{4}^{2} \frac{\Gamma(1+\alpha \alpha))^{5 \alpha}}{\Gamma^{2}(1+4 \alpha) \Gamma(1+5 \alpha)}\right]+\left[\left(\frac{u_{0}}{\Gamma(1+3 \alpha)}+\frac{u_{1}^{2}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\frac{u_{0} u_{2}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}\right) \frac{\Gamma(1+3 \alpha)}{\Gamma(1)}\right. \\
& +\left(\frac{\left.u_{0} u_{3}\right)}{\Gamma(1+4 \alpha)}+\frac{u_{1} u_{2}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}+\frac{u_{1} u_{2}}{\Gamma^{2}(1+2 \alpha)}+\frac{u_{0} u_{3}}{\Gamma(1+\alpha) \Gamma(1+3 \alpha)}\right) \frac{\Gamma(1+4 \alpha)}{\Gamma(1+\alpha)} t^{\alpha} \\
& +\left(\frac{u_{0} u_{4}}{\Gamma(1+5 \alpha)}+\frac{u_{1} u_{3}}{\Gamma(1+\alpha) \Gamma(1+4 \alpha)}+\frac{u_{2}^{2}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}+\frac{u_{1} u_{3}}{\Gamma(1+2 \alpha) \Gamma(1+3 \alpha)}\right. \\
& \left.+\frac{u_{0} u_{4}}{\Gamma(1+\alpha) \Gamma(1+4 \alpha)}\right) \frac{\Gamma(1+5 \alpha)}{\Gamma(1+2 \alpha)} t^{2 \alpha} \\
& +\left(\frac{u_{1} u_{4}}{\Gamma(1+2 \alpha) \Gamma(1+4 \alpha)}+\frac{u_{2} u_{3}}{\Gamma^{2}(1+3 \alpha)}+\frac{u_{2} u_{3}}{\Gamma(1+2 \alpha) \Gamma(1+4 \alpha)}+\frac{u_{1} u_{4}}{\Gamma(1+\alpha) \Gamma(1+5 \alpha)}\right) \frac{\Gamma(1+6 \alpha)}{\Gamma(1+3 \alpha)} t^{3 \alpha} \\
& +\left(\frac{u_{2} u_{4}}{\Gamma(1+3 \alpha) \Gamma(1+4 \alpha)}+\frac{u_{3}^{2}}{\Gamma(1+3 \alpha) \Gamma(1+4 \alpha)}+\frac{u_{2} u_{4}}{\Gamma(1+2 \alpha) \Gamma(1+5 \alpha)}\right) \frac{\Gamma(1+7 \alpha)}{\Gamma(1+4 \alpha)} t^{4 \alpha} \\
& \left.\left.+\left(\frac{u_{3} u_{4}}{\Gamma^{2}(1+4 \alpha)}+\frac{u_{3}}{\Gamma(1+3 \alpha) \Gamma(1+5 \alpha)}\right) \frac{\Gamma(1+8 \alpha)}{\Gamma(1+5 \alpha)} t^{5 \alpha}+\left(\frac{u_{4}^{2}}{\Gamma(1+4 \alpha) \Gamma(1+5 \alpha)}\right)\right) \frac{\Gamma(1+9 \alpha)}{\Gamma(1+6 \alpha)} t^{6 \alpha}\right] .
\end{aligned}
$$

Thus, we solve $D_{t}^{3 \alpha} \operatorname{Res}_{4}(0)=0$ to get

$$
\begin{aligned}
D_{t}^{3 \alpha} \operatorname{Res}_{4}(0)= & \kappa u_{4}-u_{3}+\left(2 u_{0} u_{3}+2 u_{1} u_{2} \frac{\Gamma(1+3 \alpha)}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}\right) \\
& +\left(\frac{u_{0} u_{2}}{\Gamma(1+3 \alpha)}+\frac{u_{1}^{2}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\frac{u_{0} u_{2}}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}\right) \Gamma(1+3 \alpha) \\
= & 0 .
\end{aligned}
$$

We have the coefficient $u_{4}$ as

$$
\begin{align*}
u_{4}= & \frac{1}{\kappa}\left[u_{3}-\left(2 u_{0} u_{3}+2 u_{1} u_{2} \frac{\Gamma(1+3 \alpha)}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}\right)\right.  \tag{3.52}\\
& \left.-\left(u_{0} u_{2}+\frac{u_{1}^{2} \Gamma(1+3 \alpha)}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}+\frac{u_{0} u_{2} \Gamma(1+3 \alpha)}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}\right)\right] .
\end{align*}
$$

Using a similar argument, to find $u_{k}$ in equation (3.33).
The $k^{\text {th }}$ RPS approximate solution is in form

$$
u_{k}(t)=u_{0}+\sum_{n=1}^{k} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)} .
$$

Then,

$$
\begin{align*}
u_{k}^{2}(t)= & \sum_{n=0}^{k}\left(\sum_{i=0}^{k} \frac{u_{i} u_{n-i}}{\Gamma(1+i \alpha) \Gamma(1+(n-i) \alpha)}\right) t^{n \alpha}  \tag{3.53}\\
& +\sum_{n=1}^{k}\left(\sum_{i=n}^{k} \frac{u_{i} u_{k+n-i}}{\Gamma(1+i \alpha) \Gamma(1+(k+n-i) \alpha)}\right) t^{(k+n) \alpha} .
\end{align*}
$$

We derive the $k^{\text {th }}$ residual function as

$$
\begin{aligned}
\operatorname{Res}_{k}(t)= & \kappa D_{t}^{\alpha} u_{k}(t)-u_{k}(t)+u_{k}^{2}(t)+u_{k}(t) I^{\alpha}\left(u_{k}(t)\right) \\
= & \kappa D_{t}^{\alpha}\left(u_{0}+\sum_{n=1}^{k} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)}\right)-\left(\sum_{n=0}^{k} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)}\right) \\
& {\left[\sum_{n=0}^{k}\left(\sum_{i=0}^{k} \frac{u_{i} u_{n-i}}{\Gamma(1+i \alpha) \Gamma(1+(n-i) \alpha)}\right) t^{n \alpha}\right.} \\
& \left.+\sum_{n=1}^{k}\left(\sum_{i=n}^{k} \frac{u_{i} u_{k+n-i}}{\Gamma(1+i \alpha) \Gamma(1+(k+n-i) \alpha)}\right) t^{(k+n) \alpha}\right] \\
& +\left(\sum_{n=0}^{k} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)}\right) I_{0}^{\alpha}\left(\sum_{n=0}^{k} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)}\right) \\
= & \kappa D_{t}^{\alpha}\left(u_{0}+\sum_{n=1}^{k} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)}\right)-\left(\sum_{n=0}^{k} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)}\right) \\
& {\left[\sum_{n=0}^{k}\left(\sum_{i=0}^{k} \frac{u_{i} u_{n-i}}{\Gamma(1+i \alpha) \Gamma(1+(n-i) \alpha)}\right) t^{n \alpha}\right.} \\
& \left.+\sum_{n=1}^{k}\left(\sum_{i=n}^{k} \frac{u_{i} u_{k+n-i}}{\Gamma(1+i \alpha) \Gamma(1+(k+n-i) \alpha)}\right) t^{(k+n) \alpha}\right] \\
& +\left[\sum_{n=0}^{k}\left(\sum_{i=0}^{k} \frac{u_{i} u_{n-i}}{\Gamma(1+i \alpha) \Gamma(1+(n+1-i) \alpha)}\right) t^{(n+1) \alpha}\right. \\
& \left.+\sum_{n=1}^{k}\left(\sum_{i=n}^{k} \frac{u_{i} u_{k+n-i}}{\Gamma(1+i \alpha) \Gamma(1+(k+n+1-i) \alpha)}\right) t^{(k+n+1) \alpha}\right] .
\end{aligned}
$$

So,

$$
\begin{align*}
\operatorname{Res}_{k}(t)= & \kappa \sum_{n=1}^{k} \frac{u_{n} t^{(n-1) \alpha}}{\Gamma(1+(n-1) \alpha)}-\left(\sum_{n=0}^{k} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)}\right) \\
& {\left[\sum_{n=0}^{k}\left(\sum_{i=0}^{k} \frac{u_{i} u_{n-i}}{\Gamma(1+i \alpha) \Gamma(1+(n-i) \alpha)}\right) t^{n \alpha}\right.} \\
& \left.+\sum_{n=1}^{k}\left(\sum_{i=n}^{k} \frac{u_{i} u_{k+n-i}}{\Gamma(1+i \alpha) \Gamma(1+(k+n-i) \alpha)}\right) t^{(k+n) \alpha}\right]  \tag{3.54}\\
& +\left[\sum_{n=0}^{k}\left(\sum_{i=0}^{k} \frac{u_{i} u_{n-i}}{\Gamma(1+i \alpha) \Gamma(1+(n+1-i) \alpha)}\right) t^{(n+1) \alpha}\right. \\
& \left.+\sum_{n=1}^{k}\left(\sum_{i=n}^{k} \frac{u_{i} u_{k+n-i}}{\Gamma(1+i \alpha) \Gamma(1+(k+n+1-i) \alpha)}\right) t^{(k+n+1) \alpha}\right] .
\end{align*}
$$

Now, we apply the operator $D_{t}^{(k-1) \alpha}$ on both sides of equation (3.54) becomes

$$
\begin{aligned}
D_{t}^{(k-1) \alpha} \operatorname{Res}_{k}(t)= & \kappa u_{k}-\left(u_{k-1}+\frac{u_{k} t^{\alpha}}{\Gamma(1+\alpha)}\right) \\
& +\left[\sum_{i=0}^{k-1} \frac{u_{i} u_{k-1-i} \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha)}\right. \\
& +\sum_{i=0}^{k} \frac{u_{i} u_{k-i} \Gamma(1+k \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-i) \alpha) \Gamma(1+\alpha)} t^{\alpha} \\
& \left.+\sum_{n=1}^{k}\left(\sum_{i=n}^{k} \frac{u_{i} u_{k+n-i} \Gamma(1+(k+n) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k+n-i) \alpha) \Gamma(1+(n+1) \alpha)}\right) t^{(n+1) \alpha}\right] \\
& +\left[\sum_{i=0}^{k-2} \frac{u_{i} u_{k-2-i} \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha)}\right. \\
& +\sum_{i=0}^{k-1} \frac{u_{i} u_{k-1-i} \Gamma(1+k \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-i) \alpha) \Gamma(1+\alpha)} t^{\alpha} \\
& +\sum_{i=0}^{k} \frac{u_{i} u_{k-i} \Gamma(1+(k+1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k+1-i) \alpha) \Gamma(1+2 \alpha)} t^{2 \alpha} \\
& \left.+\sum_{n=1}^{k}\left(\sum_{i=n}^{k} \frac{u_{i} u_{k+n-i} \Gamma(1+(k+n+1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k+n+1-i) \alpha) \Gamma(1+(n+2) \alpha)}\right) t^{(n+2) \alpha}\right]
\end{aligned}
$$

Solving the equation $D^{(k-1) \alpha} \operatorname{Res}_{k}(0)=0$, we have

$$
\begin{aligned}
D_{t}^{(k-1) \alpha} \operatorname{Res}_{k}(0)= & \kappa u_{k}-u_{k-1}+\left(\sum_{i=0}^{k-1} \frac{u_{i} u_{k-1-i} \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha)}\right) \\
& +\left(\sum_{i=0}^{k-2} \frac{u_{i} u_{k-2-i} \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha)}\right) \\
= & 0
\end{aligned}
$$

The coefficient $u_{k}$ is expressed as follows

$$
\begin{align*}
u_{k}= & \frac{1}{\kappa}\left[u_{k-1}-\left(\sum_{i=0}^{k-1} \frac{u_{i} u_{k-1-i} \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha)}\right)\right.  \tag{3.55}\\
& \left.-\left(\sum_{i=0}^{k-2} \frac{u_{i} u_{k-2-i} \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha)}\right)\right]
\end{align*}
$$

## Convergence analysis

In this section, we prove the convergence of the residual power series method by using Lemma 3.7.

Theorem 3.9. The fractional power series solution of fractional population growth model (3.30):

$$
u(t)=\sum_{n=0}^{\infty} \frac{u_{n} t^{n \alpha}}{\Gamma(1+n \alpha)},
$$

where the coefficients are defined in equation (3.55) has a positive radius of convergence.

Proof. Since the coefficient $u_{k}$ is

$$
\begin{aligned}
u_{k}= & \frac{1}{\kappa}\left[u_{k-1}-\left(\sum_{i=0}^{k-1} \frac{u_{i} u_{k-1-i} \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha)}\right)\right. \\
& \left.-\left(\sum_{i=0}^{k-2} \frac{u_{i} u_{k-2-i} \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha)}\right)\right] .
\end{aligned}
$$

we can see that

$$
\begin{aligned}
\frac{\left|u_{k}\right|}{\Gamma(1+k \alpha)} \leq & \left|\frac{\frac{1}{\kappa} u_{k-1}-\frac{1}{\kappa}\left(\sum_{i=0}^{k-1} \frac{u_{i} u_{k-1-i} \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha)}\right)}{\Gamma(1+k \alpha)}\right| \\
& +\left|\frac{1}{\kappa} \frac{\left(\sum_{i=0}^{k-2} \frac{u_{i} u_{k-2-i} \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha)}\right)}{\Gamma(1+k \alpha)}\right| \\
\leq & \left|\frac{1}{\kappa}\right| \frac{\left|u_{k-1}\right|}{\Gamma(1+k \alpha)}+\left|\frac{1}{\kappa}\right| \frac{\sum_{i=0}^{k-1} \frac{u_{i} u_{k-1-i} \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha)}}{\Gamma(1+k \alpha)} \\
& +\left|\frac{1}{\kappa}\right| \frac{\left|\sum_{i=0}^{k-2} \frac{u_{i} u_{k-2-i} \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha)}\right|}{\Gamma(1+k \alpha)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\left|u_{k}\right|}{\Gamma(1+k \alpha)} \leq & \left|\frac{1}{\kappa}\right| \frac{\left|u_{k-1}\right|}{\Gamma(1+k \alpha)}+\left|\frac{1}{\kappa}\right| \sum_{i=0}^{k-1} \frac{\left|u_{i}\right|\left|u_{k-1-i}\right| \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha) \Gamma(1+k \alpha)} \\
& +\left|\frac{1}{\kappa}\right| \sum_{i=0}^{k-2} \frac{\left|u_{i}\right|\left|u_{k-2-i}\right| \Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha) \Gamma(1+k \alpha)} \\
\leq & \left|\frac{1}{\kappa}\right| \frac{\left|u_{k-1}\right|}{\Gamma(1+k \alpha)} \\
& +\left|\frac{1}{\kappa}\right| \max _{0 \leq i \leq k-1}\left\{\frac{\Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha) \Gamma(1+k \alpha)}\right\} \sum_{i=0}^{k-1}\left|u_{i}\right|\left|u_{k-1-i}\right| \\
& +\left|\frac{1}{\kappa}\right| \max _{0 \leq i \leq k-2}\left\{\frac{\Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha) \Gamma(1+k \alpha)}\right\} \sum_{i=0}^{k-2}\left|u_{i}\right|\left|u_{k-2-i}\right| \\
= & A\left|u_{k-1}\right|+B \sum_{i=0}^{k-1}\left|u_{i}\right|\left|u_{k-1-i}\right|+C \sum_{i=0}^{k-2}\left|u_{i}\right|\left|u_{k-2-i}\right|,
\end{aligned}
$$

where
$A=\frac{\left|\frac{1}{\kappa}\right|}{\Gamma(1+k \alpha)}, B=\max _{0 \leq i \leq k-1}\left\{\frac{\Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha) \Gamma(1+k \alpha)}\right\}\left|\frac{1}{\kappa}\right|$ $C=\max _{0 \leq i \leq k-2}\left\{\frac{\Gamma(1+(k-1) \alpha)}{\Gamma(1+i \alpha) \Gamma(1+(k-1-i) \alpha) \Gamma(1+k \alpha)}\right\}\left|\frac{1}{\kappa}\right|$.
Let

$$
\begin{equation*}
h(t)=\sum_{k=0}^{\infty} a_{k} t^{k} \tag{3.56}
\end{equation*}
$$

where $a_{0}=\left|u_{0}\right|, a_{1}=\frac{\left|u_{1}\right|}{\Gamma(1+\alpha)}$ and

$$
\begin{equation*}
a_{k}=A a_{k-1}+B \sum_{i=0}^{k-1} a_{i} a_{k-1-i}+C \sum_{i=0}^{k-2} a_{i} a_{k-2-i}, k=2,3,4, \ldots . \tag{3.57}
\end{equation*}
$$

be the classical power series.

Thus,

$$
\begin{aligned}
\omega=h(t) & =a_{0}+a_{1} t+\sum_{k=2}^{\infty} a_{k} t^{k} \\
& =a_{0}+a_{1} t+\sum_{k=2}^{\infty}\left(A a_{k-1}+B \sum_{i=0}^{k-1} a_{i} a_{k-1-i}+C \sum_{i=0}^{k-2} a_{i} a_{k-2-i}\right) t^{k} \\
& =a_{0}+a_{1} t+A \sum_{k=2}^{\infty} a_{k-1} t^{k}+B \sum_{k=2}^{\infty}\left(\sum_{i=0}^{k-1} a_{i} a_{k-1-i}\right) t^{k}+C \sum_{k=2}^{\infty}\left(\sum_{i=0}^{k-2} a_{i} a_{k-2-i}\right) t^{k} \\
& =a_{0}+a_{1} t+A t \sum_{k=1}^{\infty} a_{k} t^{k}+B t \sum_{k=1}^{\infty}\left(\sum_{i=0}^{k} a_{i} a_{k-i}\right) t^{k}+C t^{2} \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a_{i} a_{k-i}\right) t^{k} .
\end{aligned}
$$

Let

$$
\begin{equation*}
H(t, \omega)=\omega-a_{0}-a_{1} t-A t\left(\omega-a_{0}\right)-B t\left(\omega^{2}-a_{0}^{2}\right)-C t^{2} \omega^{2} . \tag{3.58}
\end{equation*}
$$

Then

$$
H_{\omega}(t, \omega)=1-t A-2 B t \omega-2 C t^{2} \omega .
$$

Regarding at point $\left(0, a_{0}\right)$, the function $H(t, \omega)$ is 0 and the partial derivative of the function $H(t, \omega)$ with respect to $\omega$ is 1 . We can see that $H(t, \omega)$ is an analytic function, so $H(t, \omega)$ has continuous derivatives. By implicit function theorem (Rudin, 2004), there is a neighborhood of $\left(0, a_{0}\right)$ so that whenever $t$ is sufficiently close to 0 there is a unique $\omega$ so that $H(t, \omega)=0$. Then, $h(t)$ is an analytic function in the neighborhood of the point $\left(0, a_{0}\right)$ of the $(t, \omega)$-plane with a positive radius of convergence. From Lemma 3.7, the series in equation (3.32) converges.

## CHAPTER 4

## NUMERICAL RESULTS

In this chapter, we present an example to show a numerical solution by RPS method with Adomian polynomials.

## Numerical results

## Numerical results for fractional logistic equation

Firstly, we apply the RPS method for solving the fractional logistic equation. Comparison of this solution when $\alpha=1$ with the exact solution presented in equation (1.4) is reported in Tables 4.1-4.3. In the following examples, we use $k=3$ and $k=4$. Then, the numerical solution of fractional logistic equation is given by

$$
\begin{equation*}
u_{3}(t)=u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2} \alpha}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3} \alpha}{\Gamma(1+3 \alpha)} \tag{4.1}
\end{equation*}
$$

where the coefficients $u_{1}, u_{2}$ and $u_{3}$ are

$$
\begin{aligned}
& u_{1}=\rho^{\alpha}\left(u_{0}-u_{0}^{2}\right) \\
& u_{2}=\rho^{\alpha}\left(u_{1}-2 u_{0} u_{1}\right) \\
& u_{3}=\rho^{\alpha}\left(u_{2}-2 u_{0} u_{2}-u_{1}^{2} \frac{\Gamma(1+2 \alpha)}{\Gamma^{2}(1+\alpha)}\right) .
\end{aligned}
$$

Let

$$
\begin{equation*}
\operatorname{error}(t)=\left|u_{\text {exact }}(t)-u_{3}(t)\right|, t \geq 0 \tag{4.2}
\end{equation*}
$$

Example 4.1. Consider the following fractional logistic equation :

$$
D_{t}^{\alpha} u(t)=\frac{1}{2^{\alpha}} u(1-u), t>0,0<\alpha \leq 1 .
$$

with the initial condition

$$
u(0)=0.85 .
$$

Then, the error when $\alpha=1$, is reported in the following Table 4.1. Figure 4.1 shows the effect of $\alpha$ on the solution for $\alpha=0.5,0.75,1$.

Table 4.1 Error when $\alpha=1$.

| $t$ | exact solution | $u_{3}(t)$ | $\operatorname{error}(t)$ |
| :---: | :---: | :---: | :---: |
| 0.00 | 0.850000000000000 | 0.850000000000000 | 0 |
| 0.02 | 0.851270542513367 | 0.851270542493750 | $1.96169 \times 10^{-11}$ |
| 0.04 | 0.852532190262384 | 0.852532189950000 | $3.12384 \times 10^{-10}$ |
| 0.06 | 0.853784973905210 | 0.853784972331250 | $1.57396 \times 10^{-9}$ |
| 0.08 | 0.855028924550894 | 0.855028919600000 | $4.95089 \times 10^{-9}$ |
| 0.10 | 0.856264073748411 | 0.856264061718750 | $1.20297 \times 10^{-8}$ |

From Table 4.1, we can see that the approximate solution $u_{3}(t)$ is close to the exact solution.

Figure 4.1 The approximate solution of Example 4.1 for some $0<\alpha \leq 1$.

The attached Figure 4.1 illustrates the approximate solutions for various values of $0<\alpha \leq 1$.

Example 4.2. Consider the following fractional logistic equation :

$$
D_{t}^{\alpha} u(t)=\frac{1}{4^{\alpha}} u(1-u), t>0,0<\alpha \leq 1
$$

with the initial condition

$$
u(0)=\frac{1}{3} .
$$

Table 4.2 Error when $\alpha=1$.

| $t$ | exact solution | $u_{3}(t)$ | $\operatorname{error}(t)$ |
| :---: | :---: | :---: | :---: |
| 0.00 | 0.333333333333333 | 0.333333333333333 | 0 |
| 0.02 | 0.334445368823947 | 0.334445368827160 | $3.21300 \times 10^{-12}$ |
| 0.04 | 0.335559246862188 | 0.335559246913580 | $5.13920 \times 10^{-11}$ |
| 0.06 | 0.336674958073288 | 0.336674958333333 | $2.60045 \times 10^{-10}$ |
| 0.08 | 0.337792493005687 | 0.337792493827160 | $8.21437 \times 10^{-10}$ |
| 0.10 | 0.338911842131240 | 0.338911844135803 | $2.00456 \times 10^{-9}$ |

Figure 4.2 The approximate solution of Example 4.2 for some $0<\alpha \leq 1$.
The attached Figure 4.2 illustrates the approximate solutions for various values of $0<\alpha \leq 1$.
For $k=4$, the numerical solution of fractional logistic equation given by

$$
\begin{equation*}
u_{4}(t)=u_{0}+u_{1} \frac{t^{\alpha}}{\Gamma(1+\alpha)}+u_{2} \frac{t^{2} \alpha}{\Gamma(1+2 \alpha)}+u_{3} \frac{t^{3} \alpha}{\Gamma(1+3 \alpha)}+u_{4} \frac{t^{4} \alpha}{\Gamma(1+4 \alpha)} \tag{4.3}
\end{equation*}
$$

where the coefficient $u_{1}, u_{2}, u_{3}$ and $u_{4}$ are

$$
\begin{aligned}
& u_{1}=\rho^{\alpha}\left(u_{0}-u_{0}^{2}\right) \\
& u_{2}=\rho^{\alpha}\left(u_{1}-2 u_{0} u_{1}\right) \\
& u_{3}=\rho^{\alpha}\left(u_{2}-2 u_{0} u_{2}-u_{1}^{2} \frac{\Gamma(1+2 \alpha)}{\Gamma^{2}(1+\alpha)}\right) \\
& u_{4}=\rho^{\alpha}\left(u_{3}-2 u_{0} u_{3}-u_{1} u_{2} \frac{\Gamma(1+3 \alpha)}{\Gamma(1+\alpha) \Gamma(1+2 \alpha)}\right) .
\end{aligned}
$$

Let

$$
\begin{equation*}
\operatorname{error}(t)=\left|u_{\text {exact }}(t)-u_{4}(t)\right|, t \geq 0 \tag{4.4}
\end{equation*}
$$

Example 4.3. Consider the following fractional logistic equation :

$$
D_{t}^{\alpha} u(t)=\frac{1}{3^{\alpha}} u(1-u), t>0,0<\alpha \leq 1
$$

with the initial condition

$$
u(0)=\frac{3}{4}
$$

Table 4.3 Error when $\alpha=1$.

| $t$ | exact solution | $u_{4}(t)$ | error $(t)$ |
| :---: | :---: | :---: | :---: |
| 0.00 | 0.750000000000000 | 0.750000000000000 | 0 |
| 0.02 | 0.751247915518896 | 0.751247915514564 | $4.33198 \times 10^{-12}$ |
| 0.04 | 0.752491657561459 | 0.752491657492284 | $6.91750 \times 10^{-11}$ |
| 0.06 | 0.753731219529199 | 0.753731219179688 | $3.49511 \times 10^{-10}$ |
| 0.08 | 0.754966595053057 | 0.754966593950617 | $1.10244 \times 10^{-9}$ |
| 0.10 | 0.756197777992358 | 0.756197775306231 | $2.68613 \times 10^{-9}$ |

Figure 4.3 The approximate solution of Example 4.3 for some $0<\alpha \leq 1$.

The attached Figure 4.3 illustrates the approximate solutions for various values of $0<\alpha \leq 1$.

## Numerical results for the fractional Volterra population model

In this section, a numerical application of RPS method to the fractional Volterra population growth model is presented. The behaviors of approximate solutions are plotted for different values of $\alpha$, where $\alpha=\{1.0,0.75,0.5,0.25\}$, in Figure 4.4.

Example 4.4. Consider the following the fractional Volterra population growth model

$$
\begin{aligned}
\kappa D_{t}^{\alpha} u(t) & =u(t)-u^{2}(t)-u(t) I^{\alpha} u(t), \alpha \in(0,1] \\
u(0) & =0.2
\end{aligned}
$$

In this example, we use $k=6$ and $\kappa=0.3$.
Table 4.4 The solution of Example 4.4 when $k=6$.

| $t$ | $u_{6}(t)$ |
| :---: | :---: |
| 0.0 | 0.20000000000000 |
| 0.2 | 0.330324618515470 |
| 0.4 | 0.483859375387898 |
| 0.6 | 0.583645297777778 |
| 0.8 | 0.497136132370066 |
| 1.0 | 0.048529187623838 |



Figure 4.4 The approximate solution of Example 4.4 for some $0<\alpha \leq 1$.

## CHAPTER 5

## CONCLUSION AND DISCUSSION

Throughout this thesis, we study the fractional logistic equations and Voltera's population growth model. A combination of the residual power series method and the Adomian polynomials is presented for obtaining solutions from the following equations:

1. The fractional logistic equations

$$
D_{t}^{\alpha} u(t)=\rho^{\alpha} u(1-u), \alpha \in(0,1],
$$

with the initial condition

$$
u(0)=u_{0}, u_{0}>0,
$$

and $\rho>0$. The derivative in fractional logistic equation is in the Caputo sense.
2. The fractional Volterra population growth model

$$
\kappa D_{t}^{\alpha} u(t)=u(t)-u^{2}(t)-u(t) I^{\alpha} u(t), \alpha \in(0,1],
$$

with the initial condition

$$
u(0)=u_{0}, u_{0}>0,
$$

and $\kappa>0$ is a prescribed non-dimensional parameter and $u(t)$ is the scaled population of identical individuals at time $t$. The derivative in fractional Volterra population growth model is in the Caputo sense and $I^{\alpha} u(t)$ is the Riemann-Liouville fractional integral operator of order $\alpha>0$.

We have presented a combination of the residual power series method with the Adomian polynomials for obtaining solutions of fractional logistic equations and the fractional Volterra population growth model. This method provides approximate analytic solutions. The convergence analysis of the solutions was showed in Chapter 3. The numerical results of the fractional logistic equations supporting that this method is
valid with moderate accuracy. The numerical results show that this method is applicable to solve fractional Volterra population growth model. We can observe from the results that when the order $\alpha$ decreases, the critical time decreases, whereas the amplitude of $u(t)$ is consistent.

The advantages of the RPS method with Adomian polynomials are as follows:

1. We show the advantage of integrating the Adomian polynomials to the RPSM in reducing some computational work on algebraic manipulation.
2. The approximate solutions is easy to obtain from the RPS method with Adomian polynomials.
3. This method can be modified to solve the nonlinear equations.

However, a combination of the RPS method and Adomian polynomials can be applied with differential equations inclusive of the fractional differential equations especially nonlinear equations. The Adomian polynomials are used for the nonlinear term in an easy.

## Future research

1. Applying this method to solve nonlinear fractional differential equations.

## REFERENCES

Abu Arqub, O. (2013). Series solution of fuzzy differential equations under strongly generalized differentiability. Journal of Advanced Research in Applied Mathematics, 5(1), 31-52.

Abu Arqub, O., Abo-Hammour, Z., Al-Badarneh, R., \& Momani, S. (2013). A reliable analytiv method for soling high-order initial value problem. Discrete Dynamics in Nature Society, 1-12.

Adomian, G. (1984). A new approach to nonlinear partial differential equations. Journal of Mathematical Analysis and Applications, 102, 402-434.

Almeida, R., Malinowska, A. B., \& Monteiro, M. T. T. (2018). Fractional differential equations with a Caputo derivative with respect to a kernel function and their applications. Mathematical Methods in the Applied Sciences, 41(1), 336-352.

Alquran, M. (2015). Analytical solutions of fractional foam drainage equation by residual power series method. Mathematics of Sciences, 8, 153-160.

Atangana, A., \& Owolabi, K. M. (2018). New numerical approach for fractional differential equations. Mathematical Modelling of Natural Phenomena, 13(1), 1-19.

Awadalla, M., \& Yameni, Y. Y. (2018). Modeling exponential growth and exponential decay real phenomena by $\psi$-Caputo fractional derivative. Journal of Advanced in Mathematics and Computer Science, 28(2), 2456-9968.

Area, I., Losada, J., \& Nieto, J. J. (2016). A note on fractional logistic equation. Physica A., 444, 182-187.

Bhalekar, S., \& Daftardar-Gejji, V. (2012). Solving fractional-order logistic equation using a new iterative method. International Journal of Differential Equation, 1-12.

D' Ovidio, M., Loreti, P., \& Ahrabi, S. S. (2018). Modified fractional logistic equation. Physica A., 505, 818-824.

El-Ajou, A., Abu Arqub, O., Al zhour, Z., \& Momani, S. (2013). New results on fractional power series:Theories and applications. Entropy, 15, 5305-5323.

El-Sayed, A. M. A., El-mesiry, A. E. M., \& El-Saka, H. A. A. (2007). On the fractional order logistic equation. Applied mathematics letters, 20(7), 817-823.

Fatoorehchi, H. \& Abolghasemi, H. (2011). On calculation of Adomian polynomials by MATLAB. Journal of Applied Computer Science and Mathematics, 11(5), 85-88.

Foryś, U., \& Marciniak-Czochra, A. (2003). Logistic equation in tumour growth modeling. International Journal of Applied Mathematics and Computer Science, 13(3), 317-325.

Gradshteyn,I. S., \& Ryzhik, I. M. (2007). Table of Integrals, Series, and Product. San Diego: Academic Press.

Günerhan, H. (2019). Numerical method for the solution of logistic differential equations of fractional order. Turkish Journal of Analysis and Number Theory, 7(2), 33-36.

Hicdurmaz, B., \& Can, E. (2017). On the numerical solution of a fractional population growth model. Tbilisi Mathematical Journal, 10(1), 269-278.

Hilfer, R. (2000). Application of fractional calculus in physics. Singapore: World scientific publishing Co.

Jena, R. M., \& Chakraverty, S. (2019). Residual power series method for solving timefractional model of Vibration equation. Journal of Applied and Computational Mechanics, 5(4), 603-615.

Khader, M. M., \& Babatin, M. M. (2013). On approximate solutions for fractional logistic differential equation. Mathematical Problems in Engineering, 1-7.

Khader, M. M. (2016). Numerical treatment for solving fractional logistic differential equation. Differential Equations and Dynamical Systems, 24(1), 99-107.

Khan, H., Alipour, M., Khan, R. A., Tajdodi, H., \& Khan, A. (2015). On approximate solution of fractional order logistic equations by operational matrices of Bernstein polynomials. Journal of mathematics and computer science. 14, 222-232.

Kilbas, A. A., Srivastava, H. M., \& Trujillo, J. J. (2006). Theory and applications of fractional differential equations. New York : Elsevier Science Inc.

Kooi, B. W., Boer, M. P., \& Kooijman, S. A. L. (1998). On the use of the logistic equation in models of food chains. Bulletin of Mathematical Biology, $60(2)$, 231-246.

Kummer, A., Kumer, S., \& Singh, M. (2016). Residual power series method for fractional Sharma-Tasso-Olever equation. Communications in Numerical Analysis, 2016(1) 1-10.

Maleki, M., \& Kajani, M. T. (2015). Numerical approximations for Volterras population growth model with fractional order via a multi-domain pseudospectral method. Applied Mathematical Modeling, 39, 4300-4308.

Miller, K. S., \& Ross, B. (1993). An introduction to the fractional calculus and fractional differential equations. New York: John Wiley \& Sons.

Mir, Y., \& Dubeau, F. (2016). Linear and logistic model with time dependent coefficients. Electronic Journal of Differential Equations, 18, 1-17.

Mohamed, S. (2014). Application of optimal HAM for solving the fractional order logistic equation. Applied and Computational. Mathematics, 3(1), 27-31.

Momani, S., \& Qaralleh, R. (2007). Numerical approximations and Padé approximants for a fractional population growth model. Applied Mathematical Modelling, 31(9), 1907-1914.

Noupou, Y. Y. Y., Tandoǧdu, Y., \& Awadalla, M. (2019). On numerical techniques for solving the fractional logistic differential equation. Advances in Difference Equations, 1-13

Odibat, Z. M., \& Momani, S. (2008). An algorithm for the numerical solution of differential equations of fractional order. Journal of Applied Mathematics and Informatics, 26(12), 1527.

Oldhalm, K. B., \& Spainer, J. (1974). The fractional calculus. New York: Academic Press.

Parand, K., \& Delkhosh, M. (2016). Solving Volterras population growth model of arbitrary order using the generalized fractional order of the Chebyshev functions. Ricerche di Matematica, 65(1), 307-328.

Pastijn, H. (2006). In M. Ausloos \& M. Dirickx (Eds.). The Logistic Map and the Route to Chaos (pp. 3-11). Berlin: Springer-Verlag Berlin Heidelbrg.

Petráš, I. (2011). Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation. Berlin: Springer.

Petropoulou, N. E. (2010). A Discrete equivalent of logistic equation. Advance in Difference Equation, 1-15.

Podlubny, I. (1999). Fractional differential equations. San Diego: Academic Press.
Qureshi, S., Yusuf, A., Shaikh, A. A., Inc, M., \& Baleanu, D. (2019). Fractional modeling of blood ethanol concentration system with real data application. Chaos, Interdisciplinary Journal of Nonlinear Science, 29(1).

Ramesh Rao, T. R. (2018). Application of residual power series method to time fractional gas dynamics equations. Journal of Physics, 1-5.

Ramos, A. R. (2013). Logistic equation as a fractional model: It's application to business and economic. International Journal of Engineering and Applied Sciences, 2(3), 29-36.

Rudin, W. (2004). Principles of Mathematical Analysis (3 ${ }^{\text {rd }} \mathrm{ed}$ ). Beijing, China: China Machine Press.

Saad, K. M., \& Al-shomrani, A. A. (2016). Solving fractional order equation by approximate analytical methods. International Journal of Open Problems in Computer and Mathematics, 9(2), 22-36.

Samko, S. -G., Kilbas, A. -A. \& Marichev, O. -I. (1993). Fractional Integral and Derivatives, Theory and Applications, Gordon and Breach, Yverdon et alibi.

Scudo, F. M. (1971). Vito Volterra and theoretical ecology, Theoretical Population Biology, 2, 1-23.

Senol, M., \& Ata, A. (2018). Approximate solution of time-fractional KdV equations by residual power series method. Journal of Institute of Science and Technology, 20(1), 430-439.

Shatnawi, M. T. (2016). Solving boundary-layer problems by residual power series method. Journal of Mathematics Research, 8(2), 68-75.

Shoja, A., Babolian, E., \& Vahidi, A. R. (2015). The Spectral iterative method for solving fractional-order logistic equation. International of Journal Industrial Mathematics., 8(3), 215-223.

Syam, M. (2017). Analytical solution of the Fractional Fredholm integro-differntial equation using the modified residual power series method. Complexity 2017, 6, 1-6.

Syam, M. (2017). A numerical solution of fractional Lienards equation by using the residual power series method. Mathematics, 6(1), 1-9.

Sweilam, N. H., Khader, M. M., \& Mahdy, A. M. S. (2012) Numerical studies for solving fractional-order logistic equation. International Journal of Pure and Applied Mathematics, 78(8), 1199-1210.

Tariq, H., \& Akram, G. (2016). Residual power series method for solving time-spacefractional Benney-Lin arisng falling film problems. Journal of Applied Mathematics and Computing, 55, 683-708.

TeBeest, K. G. (1997). Numerical and analytic solutions of Volterras population model, Society for Industrial and Applied Mathematics, 39(3), 484-493.

Verhulst, P. F. (1838). Notice sur la loi que la population suit dans son accroissement. correspondence mathématique et physique publié par $a, 10,113-121$.

Vivek, D., Kanagarajan, K., \& Harikrishnan S. (2016). Numerical solution of fractionalorder logistic equations by fractional Euler's method. International Journal for Research in Applied Science and Engineering Technology, 4(3), 775-780.

Wazwaz, A. M. (2000). A new algorithm for calculating adomian polynomials for nonlinear operators. Applied Mathematics and Computation, 111, 53-69

West, B. J. (2015). Exact solution to fractional logistic equation. Physica A., 429, 103108.

Windarto, W., Eridaui, E., \& Purwati, U. D. (2018). A new modified logistic growth model for empirical use. Communication in Biomathematical Sciences, 1(2), 122-131.

Winley, G. K. (2007). The logistic model of growth. AU. Journal of Technology, 11(2), 99-104.

## BIOGRAPHY

| Name | Miss. Patcharee Dunnimit |
| :--- | :--- |
| Date of Birth | September 19, 1995 |
| Place of Birth | Ratchasan District, Chachoengsao Province, <br> Thailand |
| Present address | 50 Moo 1, Tambon Dongnoi, |
|  | Amphoe Ratchasan, Changwat Chachoengsao, |
|  | Thailand, 24120 |
| Education | Bachelor of Science (B. Sc.), Faculty of Science, <br> Burapha University, Chonburi, Thailand |
| $2014-2017$ | Master of Science (M. Sc.), Faculty of Science, <br> $2018-2019$ |
|  | Burapha University, Chonburi, Thailand |

