ON POLYNOMIALS GENERATED BY THE HYPER-FIBONACCI NUMBERS

JUTIPORN BOONYARAK

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF

THE REQUIREMENTS FOR MASTER OF SCIENCE

IN MATHEMATICS

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The Thesis of Jutiporn Boonyarak has been approved by the examining committee to be partial fulfillment of the requirements for the Master of Science in Mathematics of Burapha University

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In this thesis, we construct sequences of polynomials whose coefficients are hyper-Fibonacci numbers and investigate certain properties of these polynomials. In particular, we obtain results about the number of their real zeros, behavior of their complex zeros as the degree increases, and Mahler measures of these polynomials with coefficients reduced modulo Lucas numbers. Most of our results are analogous to those concerning polynomials generated by the Fibonacci sequence, which appear in work of Garth, Mills, and Mitchell.

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CHAPTER 1

INTRODUCTION

In this chapter, we give some background on polynomials generated by the Fibonacci sequence and the definition of hyper-Fibonacci numbers.

Introduction

In 2007, Garth, Mills and Mitchell began to study several properties of the zeros of polynomials which are built from the Fibonacci sequence, and they study Mahler measures of these polynomials with coefficients reduced modulo Lucas numbers. They defined the Fibonacci-coefficient polynomial (FCP) of order n, denoted by $p_n(x)$, by

$$\begin{aligned} p_n(x) &= \sum_{k=0}^{n} F_{k+1}x^{n-k} \\ &= F_1x^n + F_2x^{n-1} + F_3x^{n-2} + \cdots + F_nx + F_{n+1} \end{aligned}$$

where F_n is Fibonacci sequence defined by

$$F_n = F_{n-1} + F_{n-2}$$

for all $n \geqslant 2$, with $F_0 = 0$ and $F_1 = 1$.

For example,

$$\begin{aligned} p_1(x) &= x + 1, \\ p_2(x) &= x^2 + x + 2, \\ p_3(x) &= x^3 + x^2 + 2x + 3, \\ p_4(x) &= x^4 + x^3 + 2x^2 + 3x + 5, \\ p_5(x) &= x^5 + x^4 + 2x^3 + 3x^2 + 5x + 8, \\ p_6(x) &= x^6 + x^5 + 2x^4 + 3x^3 + 5x^2 + 8x + 13, \\ p_7(x) &= x^7 + x^6 + 2x^5 + 3x^4 + 5x^3 + 8x^2 + 13x + 21. \end{aligned}$$

Then, they studied behavior of zeros of FCPs and they found that if n is even, then FCP has no real zeros and if n is odd, FCP has only 1 real zero. Afterwards, they proved that in the latter case the sequence of unique real zeros is decreasing and approaches the negative of the golden ratio. They also proved that in general as $n \to \infty$, the zeros of FCP approach the golden ratio in modulus.

For a polynomial non-constant $P(z) = a_n z^n + \cdots + a_0 = a_n \prod_{i=1}^{n}(z - \alpha_i) \in \mathbb{C}[z]$, the Mahler measure $M(P)$ is given by

$$M(P) := |a_n| \prod_{i=1}^{n} \max(1, |\alpha_i|).$$

Garth et al. (2007) studied Mahler measures of FCPs. It is a well-known fact that for any $m \in \mathbb{N}$, the sequence of Fibonacci numbers modulo m is simply periodic.
For example,

$$\begin{aligned} F_n \mod 4 &= 0\ 1\ 1\ 2\ 3\ 1\ 0\ 1\ 1\ 2\ 3\ \ldots \\ F_n \mod 5 &= 0\ 1\ 1\ 2\ 3\ 0\ 3\ 3\ 1\ 4\ 0\ 4\ 4\ 3\ 2\ 0\ 2\ 2\ 4\ 1\ 0\ 1\ 1\ 2\ 3 \ldots \end{aligned}$$

We can adjust $F_n \mod m$ so that the residue classes range between $-\frac{m-1}{2}$ to $\frac{m-1}{2}$ when m is odd and $-\frac{m}{2} + 1$ to $\frac{m}{2}$ when m is even. For $m \geqslant 2$, they defined Fibonacci-coefficient polynomials modulo m, denoted by $p_n^{(m)}(x)$, by reducing the coefficients of $p_n(x)$ modulo m using the adjusted residue classes mentioned above. For example,

$$\begin{aligned} p_5(x) &= x^5 + x^4 + 2x^3 + 3x^2 + 5x + 8, \\ p_5^{(2)}(x) &= x^5 + x^4 + x^2 + x, \\ p_5^{(3)}(x) &= x^5 + x^4 - x^3 - x - 1. \end{aligned}$$

Finally, they consider the Mahler measure of $p_n^{(m)}(x)$ and they found that the Mahler measures of an infinite subsequence of $\{p_n^{(m)}(x)\}_{n=1}^{\infty}$ equals the Mahler measure of $p_{t-2}^{(m)}(x)$, when t is the period of the Fibonacci sequence reduced modulo m. In the particular case when $m = L_n$, the nth Lucas number, they showed that the Mahler measure of $p_n^{(L_n)}(x)$ is φ^{n-1}, where $\varphi = \dfrac{1+\sqrt{5}}{2}$.

In this thesis, we will consider hyper-Fibonacci numbers instead of Fibonacci numbers. The hyper-Fibonacci numbers $F_n^{(r)}$ can be seen as entries an infinite matrix arranged in such a way that $F_n^{(r)}$ is the entry of the $(r+1)$th row and $(n+1)$th column, satisfying $F_n^{(0)} = F_n$, $F_0^{(r)} = 0$, $F_1^{(r)} = 1$, and

$$F_n^{(r)} = F_{n-1}^{(r)} + F_n^{(r-1)}, \quad \text{for } r \geqslant 1, n \geqslant 1.$$

We define hyper-Fibonacci-coefficient polynomial (hFCP) as

$$\begin{aligned} p_{r,n}(x) &= \sum_{k=0}^{n} F_{k+1}^{(r)} x^{n-k} \\ &= F_1^{(r)} x^n + F_2^{(r)} x^{n-1} + F_3^{(r)} x^{n-2} + \cdots + F_n^{(r)} x + F_{n+1}^{(r)}. \end{aligned}$$

We study zeros of hFCP's. In particular, we expect that they have several properties analogous to those of FCPs. Finally, we plan to study period of hyper-Fibonacci numbers for some r.

Research objectives

1. To study hyper-Fibonacci numbers.
2. To investigate the existence and bounds of real zeros of hFCP's .
3. To study behavior of moduli of zeros of some hFCP's $p_{r,n}(x)$ for some r and large n.
4. To study the period of hyper-Fibonacci numbers $F_n^{(r)}$ for some r.

Scope of the study

In this thesis, we will study hFCPs defined by

$$\begin{aligned} p_{r,n}(x) &= \sum_{k=0}^{n} F_{k+1}^{(r)} x^{n-k} \\ &= F_1^{(r)} x^n + F_2^{(r)} x^{n-1} + F_3^{(r)} x^{n-2} + \cdots + F_n^{(r)} x + F_{n+1}^{(r)} \end{aligned}$$

and investigate their properties in directions analogous to those of Garth et al. (2007). In particular, we determine location of their real zeros and the behavior of zeros of $p_{r,n}(x)$ as $n \to \infty$. Finally, we will try to understand periodicity of hyper-Fibonacci numbers for some r. In the initial stage of our study, we consider the case $r = 1$. If possible, we will try to extend our results to arbitrary $r \in \mathbb{N} \cup \{0\}$.

CHAPTER 2

PRELIMINARIES AND LITERATURE REVIEWS

Fibonacci sequence

Definition 2.1. *(Koshy, 2011) The Fibonacci numbers, denoted by F_n, are defined by*

$$F_n = F_{n-1} + F_{n-2} \quad \textit{for} \quad n \geqslant 1$$

for all $n \geqslant 2$, with $F_0 = 0$ and $F_1 = 1$.

Definition 2.2. *(Garth, Mills & Mitchell, 2007) The Fibonacci-coefficient polynomial (FCP) of order n, denoted by $p_n(x)$, is defined by*

$$\begin{aligned} p_n(x) &= \sum_{k=0}^{n} F_{k+1}x^{n-k} \\ &= F_1x^n + F_2x^{n-1} + F_3x^{n-2} + \cdots + F_nx + F_{n+1}. \end{aligned}$$

Example 2.3.

$$\begin{aligned} p_1(x) &= x + 1, \\ p_2(x) &= x^2 + x + 2, \\ p_3(x) &= x^3 + x^2 + 2x + 3, \\ p_4(x) &= x^4 + x^3 + 2x^2 + 3x + 5, \\ p_5(x) &= x^5 + x^4 + 2x^3 + 3x^2 + 5x + 8, \\ p_6(x) &= x^6 + x^5 + 2x^4 + 3x^3 + 5x^2 + 8x + 13, \\ p_7(x) &= x^7 + x^6 + 2x^5 + 3x^4 + 5x^3 + 8x^2 + 13x + 21. \end{aligned}$$

Definition 2.4. *(Garth et al., 2007) Fibonacci-coefficient polynomials modulo m, denoted by $p_n^{(m)}(x)$, is constructed by reducing the coefficients of $p_n(x)$ modulo m using*

$$(F_k \mod m) - m \quad \textit{if} \quad (F_k \mod m) > \frac{m}{2}.$$

Example 2.5. *The Fibonacci-coefficient polynomial (FCP) of order 4 modulo m*

$$\begin{aligned} p_4(x) &= x^4 + x^3 + 2x^2 + 3x + 5, \\ p_4^{(2)}(x) &= x^4 + x^3 + x + 1, \\ p_4^{(3)}(x) &= x^4 + x^3 - x^2 - 1, \\ p_4^{(4)}(x) &= x^4 + x^3 + 2x^2 - x + 1, \\ p_4^{(5)}(x) &= x^4 + x^3 + 2x^2 - 2x. \end{aligned}$$

Example 2.6. *The Fibonacci-coefficient polynomial (FCP) of order 5 modulo m*

$$\begin{aligned} p_5(x) &= x^5 + x^4 + 2x^3 + 3x^2 + 5x + 8, \\ p_5^{(2)}(x) &= x^5 + x^4 + x^2 + x, \\ p_5^{(3)}(x) &= x^5 + x^4 - x^3 - x - 1, \\ p_5^{(4)}(x) &= x^5 + x^4 + 2x^3 - x^2 + x, \\ p_5^{(5)}(x) &= x^5 + x^4 + 2x^3 - 2x^2 - 2. \end{aligned}$$

Example 2.7. *The Fibonacci-coefficient polynomial (FCP) of order 6 modulo m*

$$\begin{aligned} p_6(x) &= x^6 + x^5 + 2x^4 + 3x^3 + 5x^2 + 8x + 13, \\ p_6^{(2)}(x) &= x^6 + x^5 + x^3 + x^2 + 1, \\ p_6^{(3)}(x) &= x^6 + x^5 - x^4 - x^2 - x + 1, \\ p_6^{(4)}(x) &= x^6 + x^5 + 2x^4 - x^3 + x^2 + 1, \\ p_6^{(5)}(x) &= x^6 + x^5 + 2x^4 - 2x^3 - 2x - 2. \end{aligned}$$

Definition 2.8. *(Koshy, 2011) The Lucas numbers, denoted L_n, are defined by*

$$L_n = L_{n-1} + L_{n-2} \quad \textit{for} \quad n \geqslant 1$$

for all $n \geqslant 2$, with $L_0 = 2$ and $L_1 = 1$.

The following formula gives a close relationship between the Fibonacci numbers, the Lucas numbers and the golden ratio $\varphi = \dfrac{1+\sqrt{5}}{2}$.

Lemma 2.9. *(Koshy, 2011) (Binet's Formula) Let $\varphi = \dfrac{1+\sqrt{5}}{2}$ and $\tau = \dfrac{1-\sqrt{5}}{2}$. For all $n \in \mathbb{N}$, we have*

$$F_n = \frac{\varphi^n - \tau^n}{\varphi - \tau} = \frac{1}{\sqrt{5}}\varphi^n + \frac{1}{\sqrt{5}} \cdot \left(-\frac{1}{\varphi}\right)^n$$

and

$$L_n = \varphi^n + \tau^n = \varphi^n + \left(-\frac{1}{\tau}\right)^n.$$

Periods of Fibonacci sequence reduced modulo m

For any integer $m \geqslant 2$, the Fibonacci sequence becomes periodic when reduced modulo m. In general the Fibonacci sequence modulo m must repeat because there are only m^2 possible pairs of residue classes. Note that any two consecutive Fibonacci numbers cannot be both zero modulo m, so the period of any Fibonacci sequence mod m has a maximum length of $m^2 - 1$ (Wall, 1960).

Example 2.10.

$$\begin{aligned}
F_n \mod 3 &= \mathbf{0\ 1\ 1\ 2\ 0\ 2\ 2\ 1}\ 0\ 1\ 1\ 2\ \ldots \\
F_n \mod 4 &= \mathbf{0\ 1\ 1\ 2\ 3\ 1}\ 0\ 1\ 1\ 2\ 3\ \ldots \\
F_n \mod 5 &= \mathbf{0\ 1\ 1\ 2\ 3\ 0\ 3\ 3\ 1\ 4\ 0\ 4\ 4\ 3\ 2\ 0\ 2\ 2\ 4\ 1}\ 0\ 1\ 1\ 2\ 3\ldots \\
F_n \mod 7 &= \mathbf{0\ 1\ 1\ 2\ 3\ 4\ 1\ 5\ 1\ 6\ 0\ 6\ 6\ 5\ 4\ 2\ 6\ 1}\ 0\ 1\ 1\ 2\ 3\ \ldots \\
F_n \mod 11 &= \mathbf{0\ 1\ 1\ 2\ 3\ 5\ 8\ 2\ 10\ 1}\ 0\ 1\ 1\ 2\ 3\ 5\ \ldots
\end{aligned}$$

Lemma 2.11. *(Garth et al., 2007) If n is odd, then the period of the Fibonacci sequence modulo L_n, with residue classes adjusted to range between $-\frac{L_n - 1}{2}$ to $\frac{L_n - 1}{2}$ for odd L_n and $-\frac{L_n}{2} + 1$ to $\frac{L_n}{2}$ for even L_n, is*

$$F_0,\ F_1,\ F_2,\ F_3,\ \ldots,\ F_n,\ (-F_{n-1}),\ F_{n-2},\ (-F_{n-3}),\ F_{n-4},\ \ldots,\ (-F_2),\ F_1.$$

Example 2.12. $F_n \Rightarrow$ *0 1 1 2 3 5 8 13 21 34 55 89 144* $\ldots$

$F_k \mod L_3 \;\Rightarrow\; F_k \mod 4$ *we have range between -1 to 2.*

0 1 1 2 -1 1 0 1 1 2 -1 1 0 1 1 2 -1 1 $\ldots$

Thus period is *0 1 1 2 -1 1*

F_0 F_1 F_2 F_3 -F_2 F_1.

Example 2.13. $F_n \Rightarrow$ *0 1 1 2 3 5 8 13 21 34 55 89 144* $\ldots$

$F_k \mod L_5 \;\Rightarrow\; F_k \mod 11$ *we have range between -5 to 5.*

0 1 1 2 3 5 -3 2 -1 1 0 1 1 2 3 5 -3 2 -1 1 0 1 $\ldots$

Thus period is *0 1 1 2 3 5 -3 2 -1 1*

F_0 F_1 F_2 F_3 F_4 F_5 -F_4 F_3 - F_2 F_1.

Lemma 2.14. *(Garth et al., 2007) If n is even, then the period of the Fibonacci sequence modulo L_n with residue classes adjusted to range between $-\frac{L_n - 1}{2}$ to $\frac{L_n - 1}{2}$ for odd L_n and $-\frac{L_n}{2} + 1$ to $\frac{L_n}{2}$ for even L_n, is*

$$F_0,\ F_1,\ F_2,\ F_3,\ \ldots,\ F_n,\ (-F_{n-1}),\ F_{n-2},\ (-F_{n-3}),\ F_{n-4},\ \ldots,\ F_2,\ (-F_1),$$

$$F_0,\ (-F_1),\ (-F_2),\ (-F_3),\ \ldots,\ (-F_n),\ F_{n-1},\ (-F_{n-2}),\ F_{n-3},\ \ldots,\ (-F_2),\ F_1.$$

Example 2.15. $F_n \Rightarrow$ *0 1 1 2 3 5 8 13 21 34 55 89 144* $\ldots$

$F_k \mod L_2 \;\Rightarrow\; F_k \mod 3$ *we have range between -1 to 1.*

0 1 1 -1 0 -1 -1 1 0 1 1 -1 0 -1 -1 1 0 1 $\ldots$

Thus period is *0 1 1 -1 0 -1 -1 1*

F_0 F_1 F_2 -F_1 F_0 -F_1 -F_2 F_1.

Example 2.16. $F_n \Rightarrow$ *0 1 1 2 3 5 8 13 21 34 55 89 144* $\ldots$

$F_k \mod L_4 \;\Rightarrow\; F_k \mod 7$ *we have range between -3 to 3.*

0 1 1 2 3 -2 1 -1 0 -1 -1 -2 -3 2 -1 1 0 1 $\ldots$

Thus period is

0	*1*	*1*	*2*	*3*	*-2*	*1*	*-1*	*0*	*-1*	*-1*	*-2*	*-3*	*2*	*-1*	*1*
F_0	F_1	F_2	F_3	F_4	-F_3	F_2	- F_1	F_0	-F_1	-F_2	-F_3	-F_4	F_3	-F_2	F_1.

Zeros of FCPs

Garth et al. (2007) investigated the zeros of the FCPs and they obtained several results, as given below.

Let $g_n(x) = p_n(x)(x^2 - x - 1)$ for $n \in \mathbb{N}$. In the proofs of their main results, they used the following identities, which are straightforward to verify.

$$\begin{aligned} g_n(x) := p_n(x)(x^2 - x - 1) &= x^{n+2} - F_{n+2}x - F_{n+1}, \\ x^{n+1} - F_{n+1} &= (x-1)p_n(x) - xp_{n-2}(x), \\ x^n + F_n &= p_n(x) - (x+1)p_{n-2}(x). \end{aligned}$$

Theorem 2.17. *(Garth et al., 2007) For even n, $p_n(x)$ has no real zeros. For odd n, $p_n(x)$ has exactly one real zero, which lies in the interval $(-\varphi, -1]$. Moreover, the sequence of real zeroes of the $p_n(x)$, with n odd, decreases monotonically to $-\varphi$.*

Example 2.18.

$$p_4(x) = x^4 + x^3 + 2x^2 + 3x + 5$$

Zeros of this polynomial are (approximately) 0.5784 ± 1.4629i, -1.0784 ± 0.9260i, so $p_4(x)$ has no real zeros.

Example 2.19.

$$p_5(x) = x^5 + x^4 + 2x^3 + 3x^2 + 5x + 8$$

Zeros of this polynomial are (approximately) 0.8556 ± 1.3422i, -1.3912, -0.6601 ± 1.3543i,

so $p_5(x)$ has exactly one real zero, namely -1.3912... $\in (-\varphi, -1]$.

The proof of Theorem 2.17 relies on Descartes' rule of sign and the intermediate value theorem. Descartes' rule of sign can be used to estimate the number of positive real zeros and negative real zeros by counting the sign changes of the coefficients of the polynomial function $f(x)$ and $f(-x)$, respectively (Anderson, Jackson & Sitharam, 2018).

Example 2.20.

$$f(x) = 2x^5 - x^4 + 5x^3 + 6x^2 - x + 8$$

For the positive-root case, consider

$$f(x) = 2x^5 - x^4 + 5x^3 + 6x^2 - x + 8.$$

There are four sign changes in $f(x)$. This number "four" is the maximum possible number of positive zeros of this polynomial.

For the negative-root case, consider

$$\begin{aligned} f(-x) &= 2(-x)^5 - (-x)^4 + 5(-x)^3 + 6(-x)^2 - (-x) + 8 \\ &= -2x^5 - x^4 - 5x^3 + 6x^2 + 2x + 8. \end{aligned}$$

There is only one sign change in $f(-x)$, so $f(x)$ has at most one negative root. Therefore there are 4, 2, or 0 positive roots, and exactly 1 negative root.

Example 2.21.

$$f(x) = 4x^7 + 3x^6 + x^5 + 2x^4 - x^3 + 9x^2 + x + 1$$

For the positive-root case, consider

$$f(x) = 4x^7 + 3x^6 + x^5 + 2x^4 - x^3 + 9x^2 + x + 1.$$

There are two sign changes in $f(x)$. This number "two" is the maximum possible number of positive zeroes of this polynomial.
For the negative-root case, consider

$$\begin{aligned} f(-x) &= 4(-x)^7 + 3(-x)^6 + (-x)^5 + 2(-x)^4 - (-x)^3 + 9(-x)^2 + (-x) + 1 \\ &= -4x^7 + 3x^6 - x^5 + 2x^4 + x^3 + 9x^2 - x + 1. \end{aligned}$$

There are five sign changes in $f(x)$. This number "five" is the maximum possible number of negative zeroes of this polynomial .

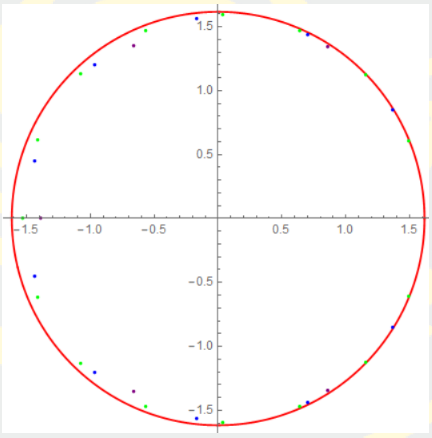
Therefore there are 2, or 0 positive roots, and 5, 3, or 1 negative roots.

Theorem 2.22. *(Bartle & Sherbert, 2000) (Intermediate value theorem) If f is a continuous function in the interval [a,b] and c is arbitrary between $f(a)$ and $f(b)$, then there exists x in the interval [a,b] such that $f(x) = c$.*

Theorem 2.23. *(Garth et al., 2007) As n increases without bound, the roots of $p_n(x)$ approach φ in modulus.*

We use Mathematica to locate the zeros of FCP of order 5, 10, and 15, as illustrated in the figure below.

Figure 2.1 The zeros of $p_n(x)$ with $n = 5, 10, 15$



In the proof of Theorem 2.23, Garth et al. use the following result:

Theorem 2.24. *(Marden, 1966) (Rouché's Theorem) If $P(z)$ and $Q(z)$ are analytic interior to a simple closed Jordan curve C, and if they are continuous on C and*

$$|P(z)| < |Q(z)|, \quad z \in C,$$

then the function $H(z) = P(z) + Q(z)$ has the same number of zeros interior to C as does $Q(z)$.

Mahler Measures

Definition 2.25. *(Mossinghoff, 1998) Let $P(z) = a_n z^n + \cdots + a_0 = a_n \prod_{i=1}^{n} (z - \alpha_i) \in \mathbb{C}[z]$ be a non-constant polynomial.The Mahler Measure M(P) is given by*

$$\begin{aligned} M(P) &:= |a_n| \prod_{i=1}^{n} \max(1, |\alpha_i|) \\ &= |a_n| \prod_{|\alpha_i|>1} |\alpha_i|. \end{aligned}$$

Garth et al. (2007) consider the Mahler measure of the $p_k^{(m)}(x)$ defined in the introduction. They consider the Mahler measures of an infinite subsequence of $\left\{p_n^{(m)}(x)\right\}_{n=1}^{\infty}$ and prove the following theorem:

Theorem 2.26. *(Garth et al., 2007) Let $m \geqslant 2$, and let t be the number of terms in one period of the Fibonacci sequence reduced modulo m. If $k \equiv -2$ or $-1 \mod t$, then*

$$M\left(p_k^{(m)}(x)\right) = M\left(p_{t-2}^{(m)}(x)\right).$$

Finally, Garth et al. (2007) established the following theorem:

Theorem 2.27. *(Garth et al., 2007) If $k \equiv -2$ or $-1 \mod 2n$ then $M\left(p_k^{(L_n)}(x)\right) = \varphi^{n-1}$.*

hyper-Fibonacci numbers

Dil and Mezó (2008) defined hyper-Fibonacci numbers $F_n^{(r)}$ as follows:

Definition 2.28. *(Dil & Mezó, 2008) The hyper-Fibonacci numbers $F_n^{(r)}$ are defined by the following recursive relation:*

$$F_n^{(r)} = \sum_{k=0}^{n} F_k^{(r-1)}, \ \ with \ \ F_n^{(0)} = 0, \ \ and \ \ F_1^{(r)} = 1.$$

Table 2.1 Sequence of hyper-Fibonacci numbers in the first few generations

n	0	1	2	3	4	5	6	7	$\cdots$
$F_n^{(0)}$	0	1	1	2	3	5	8	13	$\cdots$
$F_n^{(1)}$	0	1	2	4	7	12	20	33	$\cdots$
$F_n^{(2)}$	0	1	3	7	14	26	46	79	$\cdots$
$F_n^{(3)}$	0	1	4	11	25	51	97	179	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

We record the generating function and a useful identity for $F_n^{(r)}$ below.

Proposition 2.29. *(Dil & Mezó, 2008) The generating function of the hyper-Fibonacci numbers can be written as*

$$\sum_{n=0}^{\infty} F_n^{(r)} t^n = \frac{t}{(1-t-t^2)(1-t)^r}.$$

Lemma 2.30. *(Cristea, Martinjak & Urbiha, 2016) The difference between the n-th r-generation hyperfibonacci number and the sum of its two consecutive predecessors is the n-th regular (r-1)-topic number; i.e.,*

$$F_{n+2}^{(r)} = F_{n+1}^{(r)} + F_n^{(r)} + \binom{n+r}{r-1}, \quad n \geqslant 0.$$

Literature Reviews

Mátyás (2007) considered the nth term of the second order linear recursive sequence

$$R = \{R_n\}_{n=0}^{\infty},$$

where $n \geqslant 2$, $R_0 = 0$ and $R_1 = 1$

$$R_n = AR_{n-1} + BR_{n-2}.$$

In special case $A = B = 1$, R_n is the usual Fibonacci-sequence

$$F_n = F_{n-1} + F_{n-2} \quad (n \geqslant 2) \qquad \text{where} \;\; F_0 = 0, F_1 = 1.$$

Following work of Garth et al. (2007), Mátyás (2007) studied the zeros of the generalized Fibonacci-coefficient polynomial (GFCP) $q_n(x)$ defined by

$$\begin{aligned} q_n(x) &= \sum_{k=0}^{n} R_{k+1}x^{n-k} \\ &= R_1x^n + R_2x^{n-1} + R_3x^{n-2} + \cdots + R_nx + R_{n+1}. \end{aligned}$$

Later, Mátyás (2008) studied the zeros of much more general polynomials $q_n^{(i)}(x)$ and $q_n^{(i,t)}(x)$, defined by

$$\begin{aligned} q_n^{(i)}(x) &= \sum_{k=0}^{n} R_{i+k}x^{n-k} \\ &= R_ix^n + R_{i+1}x^{n-1} + R_{i+2}x^{n-2} + \cdots + R_{i+n-1}x + R_{i+n}, \end{aligned}$$

$$\begin{aligned} q_n^{(i,t)}(x) &= \sum_{k=0}^{n} R_{i+kt}x^{n-k} \\ &= R_ix^n + R_{i+t}x^{n-1} + R_{i+2t}x^{n-2} + \cdots + R_{i+(n-1)t}x + R_{i+nt}, \end{aligned}$$

Mátyás and Szalay (2011) studied the Tribonacci sequence defined by $T_n = T_{n-1} + T_{n-2} + T_{n-3}$, $n \geqslant 3$ with initial values $T_0 = 0$, $T_1 = 0$ and $T_2 = 1$, and they considered the Tribonacci-coefficient polynomial

$$P_n(x) = T_2x^n + T_3x^{n-1} + \cdots + T_{n+1}x + T_{n+2}.$$

For even $n \geqslant 2$, they showed that $q_n(x)$, $q_n^{(i)}(x)$, $q_n^{(i,t)}(x)$ and $P_n(x)$ have no real zeros, and for odd $n \geqslant 3$, $q_n(x)$, $q_n^{(i)}(x)$, $q_n^{(i,t)}(x)$ and $P_n(x)$ have exactly one real zero.

Mansour and Shattuck (2012) studied the k-Fibonacci numbers a_n defined by

$$a_n = a_{n-1} + a_{n-2} + \cdots + a_{n-k}, \quad (n \geqslant k)$$

with the initial values $a_0 = a_1 = \cdots = a_{k-2} = 0$ and $a_{k-1} = 1, \quad k \geqslant 2$. These reduce to the Fibonacci numbers F_n when $k = 2$ and Tribonacci numbers T_n when $k = 3$. Then they defined the k-Fibonacci coefficient polynomial $P_{n,k}(x)$ by

$$P_{n,k}(x) = a_{k-1}x^n + a_k x^{n-1} + \cdots + a_{n+k-2}x + a_{n+k-1}.$$

They obtain the following result:

If n is even, then $P_{n,k}(x)$ has no real zeros, and if n is odd, then $P_{n,k}(x)$ has exactly one real zero, denoted by $r_{n,k}$. Moreover, they proved that if $k \geqslant 2$ and n is odd, then $r_{n,k} \to -\lambda$ as $n \to \infty$, where λ is the unique real zero of $c_k(x) = x^k - x^{k-1} - \cdots - x - 1$ such that $\lambda > 1$.

In 2017, Sitthaset, Laohakosol, and Mavecha extended the results of Garth, Mills and Mitchell (2007) to those of a generalized Fibonacci-coefficient polynomial (GFCP), initially studied by Mátyás (2007). They obtain results similarly to those of Garth, Mills and Mitchell (2007) as follows :

For even $n \geqslant 2$, $q_n(x)$ has no real zeros, and for odd $n \geqslant 3$, $q_n(x)$ has exactly one real zero. They proved in addition that if $A^2 \geqslant B$, then this zero lies in the interval $(-\varphi_{A,B}, -A]$, where $\varphi_{A,B} = \dfrac{A + \sqrt{A^2 + 4B}}{2}$. Furthermore, the sequence of real zeros of the polynomials $q_n(x)$ with odd n converges to $-\varphi_{A,B}$ and the roots of $q_n(x)$ approach $\varphi_{A,B}$ in modulus as $n \to \infty$. Let $m \geqslant 2$ and t be the number of terms in one period of generalized Fibonacci sequence modulo m. Sitthaset et al. (2017) showed that if $k \equiv -2$ or $-1(\mod t)$, then the Mahler measure of $q_k^{(m)}(x)$ equals Mahler measure of $q_{t-2}^{(m)}(x)$.

Finally, they proved that if $k \equiv -2$ or $-1 \mod 2n$, then $M(q_k^{(L_n)}(x)) = -\varphi_{A,B}^{n-1}$, where L_n is the nth Lucas number.

CHAPTER 3

RESEARCH METHODOLOGY

In this thesis, we will study hyper-Fibonacci-coefficient polynomials (hFCP). We do the following process.

1. We compute the zeros of the $p_{r,n}(x)$ numerically and predict the number of real zeros of $p_{r,n}(x)$ and the behavior of all zeros as $n \to \infty$ using MATLAB.

2. We make conjectures based on our numerical results.

3. We prove our conjectures and make a conclusion.

CHAPTER 4

POLYNOMIALS GENERATED BY HYPER-FIBONACCI NUMBERS

We have investigated the zeros of $p_{1,n}(x)$. We prove that if n is even, then $p_{1,n}(x)$ has no real zeros and if n is odd, then $p_{1,n}(x)$ has a unique real zero which lies in the interval $[-2, -\varphi)$. After that, we show that the complex zeros of $p_{1,n}(x)$ approach φ in modulus as $n \to \infty$. Finally, we have studied Mahler Measures of $p_{1,n}(x)$ whose coefficients are reduced modulo $m \in \mathbb{N}$, which we denote by $p_{1,n}^{(m)}(x)$. In this chapter we will give detailed proofs of our results and give suggestions for future research.

4.1 Zeros of hFCPs

Before stating and proving our main results about zeros of hFCPs, let us introduce the auxiliary polynomials $g_{r,n}(x)$ which are defined as follows:

$$g_{r,n}(x) := p_{r,n}(x)(x^2 - x - 1)(x - 1)^r, \quad r, n \in \mathbb{N} \cup \{0\}. \tag{4.1}$$

It is clear from (4.1) that the zeros of $g_{r,n}(x)$ completely determines those of $p_{r,n}(x)$ and we will focus on $g_{r,n}(x)$ first. The expansion of $g_{0,n}(x)$ has been given in (Garth, Mills & Mitchell, 2007), namely

$$g_{0,n}(x) = x^{n+2} - F_{n+2}x - F_{n+1}. \tag{4.2}$$

The expansions of $g_{r,n}(x)$ for the first few values of $r \geqslant 1$ are given below.

Example 4.1.

$$
\begin{aligned}
g_{1,n}(x) &= x^{n+3} - (F_{n+1}^{(1)} + F_n^{(1)} + 1)x^2 + F_n^{(1)}x + F_{n+1}^{(1)},\\
g_{2,n}(x) &= x^{n+4} - (3F_{n+1}^{(2)} - 2F_n^{(2)} - F_{n-2}^{(2)})x^3 + (2F_{n+1}^{(2)} + F_n^{(2)} - F_{n-1}^{(2)})x^2\\
&\qquad + (F_{n+1}^{(2)} - F_n^{(2)})x - F_{n+1}^{(2)},\\
g_{3,n}(x) &= x^{n+5} - (4F_{n+1}^{(3)} - 5F_n^{(3)} + F_{n-1}^{(3)} + 2F_{n-2}^{(3)} - F_{n-3}^{(3)})x^4\\
&\qquad + (5F_{n+1}^{(3)} - F_n^{(3)} - 2F_{n-1}^{(3)} + F_{n-2}^{(3)})x^3 - (F_{n+1}^{(3)} + 2F_n^{(3)} - F_{n-1}^{(3)})x^2\\
&\qquad - (2F_{n+1}^{(3)} - F_n^{(3)})x + F_{n+1}^{(3)}.
\end{aligned}
$$

For $r \geq 1$, we have the following expression of $g_{r,n}(x)$.

Proposition 4.2. *For $r, n \in \mathbb{N} \cup \{0\}$ with $r \geq 1$, we have*

$$
\begin{aligned}
g_{r,n}(x) &= x^{r+n+2} - F_{n+2}x^{r+1} - F_{n+1}x^r\\
&\qquad + (-1)^r(x^2 - x - 1)\sum_{m=0}^{r-1}\left(\sum_{p=0}^{n}(-1)^{m-p}\binom{r}{m-p}F_{n-p+1}^{(r)}\right)x^m.
\end{aligned} \tag{4.3}
$$

Observe from (4.3) that $g_{r,n}(x)$ has at most $r+3$ nonzero coefficients, since the largest power of x in the product of $x^2 - x - 1$ and the double summation is x^{r+1}. To prove (4.3), we need the following lemmas.

Lemma 4.3. *For $r, l \in \mathbb{N} \cup \{0\}$, we have*

$$F_{l+1} = \sum_{k=0}^{l}(-1)^k\binom{r}{k}F_{l-k+1}^{(r)}.$$

Proof. Recall from Proposition 2.29 that the generating function of the hyper-Fibonacci numbers can be written as

$$\sum_{n=0}^{\infty}F_n^{(r)}t^n = \frac{t}{(1-t-t^2)(1-t)^r}.$$

Therefore, we have that for any $r \in \mathbb{N} \cup \{0\}$

$$\sum_{m=0}^{\infty}F_{m+1}^{(r)}t^m = \frac{1}{t}\sum_{m=0}^{\infty}F_m^{(r)}t^m = \frac{1}{(1-t-t^2)(1-t)^r}.$$

It follows that

$$\begin{aligned}\sum_{l=0}^{\infty} F_{l+1}t^l &= \frac{1}{1-t-t^2} \\ &= (1-t)^r \sum_{m=0}^{\infty} F_{m+1}^{(r)} t^m \\ &= \left(\sum_{k=0}^{\infty}(-1)^k \binom{r}{k} t^k\right)\left(\sum_{m=0}^{\infty} F_{m+1}^{(r)} t^m\right) \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(-1)^k \binom{r}{k} F_{m+1}^{(r)} t^{k+m} \\ &= \sum_{l=0}^{\infty}\left(\sum_{k=0}^{l}(-1)^k \binom{r}{k} F_{l-k+1}^{(r)}\right) t^l.\end{aligned}$$

The proof is finished by comparing the coefficients of both sides. □

Lemma 4.4. *For $r, n \in \mathbb{N} \cup \{0\}$ with $r \geqslant 1$, let*

$$A = \{(m,p) \in \{0,1,\ldots,r-1\} \times \{0,1,\ldots,\min\{m,n\}\}\}$$
$$B = \{(k,j) \in \{0,1,\ldots,r\} \times \{0,1,\ldots,n\} \mid n+1 \leqslant k+j \leqslant n+r\}.$$

Then the assignment $f : A \to B$ given by $f(m,p) = (r-(m-p), n-p)$ is a one-to-one correspondence.

Proof. We will show first that f is well-defined.

Let $(m,p) \in A$. Since $0 \leqslant p \leqslant \min\{m,n\}$, we have $0 \leqslant r-(m-p) \leqslant r$ and $n-p \in \{0,1,\ldots,n\}$. Therefore, $(r-(m-p), n-p) \in \{0,1,\ldots,r\} \times \{0,1,\ldots,n\}$.

Since $1 \leqslant r-m \leqslant r$, it follows that $n+1 \leqslant r-m+n \leqslant n+r$, so $f(m,p) \in B$.

Next, we show that f is injective.

Let $(a,b),(c,d) \in A$ and suppose that $f(a,b) = f(c,d)$.

Then we have $(r-(a-b), n-b) = (r-(c-d), n-d)$. It follows immediately that $b = d$ and $a = c$, so $(a,b) = (c,d)$.

Finally, we show that f is surjective.

Let $(k,j) \in B$. Then $0 \leqslant k \leqslant r$, $0 \leqslant j \leqslant n$ and $n+1 \leqslant k+j \leqslant n+r$.

Hence $0 \leqslant n+r-k-j \leqslant r-1$ and $0 \leqslant n-j \leqslant n+r-k-j$.

Therefore, $(n+r-k-j, n-j) \in A$ and $f(n+r-k-j, n-j) = (k, j)$. $\square$

Proof of Proposition 4.2. Let $r \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$. Then by the definition of $p_{r,n}(x)$ and the binomial theorem we have

$$\begin{aligned}
(x-1)^r p_{r,n}(x) - x^r p_{0,n}(x) &= \sum_{k=0}^{r}\sum_{j=0}^{n}(-1)^k \binom{r}{k} F_{j+1}^{(r)} x^{n+r-k-j} - \sum_{l=0}^{n} F_{l+1} x^{n+r-l} \\
&= \sum_{\substack{0\le k\le r\\ 0\le j\le n\\ n+1\le k+j\le n+r}} (-1)^k \binom{r}{k} F_{j+1}^{(r)} x^{n+r-k-j} \\
&\quad + \sum_{\substack{0\le k\le r\\ 0\le j\le n\\ 0\le k+j\le n}} (-1)^k \binom{r}{k} F_{j+1}^{(r)} x^{n+r-k-j} - \sum_{l=0}^{n} F_{l+1} x^{n+r-l} \\
&= \sum_{\substack{0\le k\le r\\ 0\le j\le n\\ n+1\le k+j\le n+r}} (-1)^k \binom{r}{k} F_{j+1}^{(r)} x^{n+r-k-j} \\
&\quad + \sum_{l=0}^{n}\sum_{k=0}^{l}(-1)^k \binom{r}{k} F_{l-k+1}^{(r)} x^{n+r-l} - \sum_{l=0}^{n} F_{l+1} x^{n+r-l} \\
&= \sum_{\substack{0\le k\le r\\ 0\le j\le n\\ n+1\le k+j\le n+r}} (-1)^k \binom{r}{k} F_{j+1}^{(r)} x^{n+r-k-j},
\end{aligned}$$

where the last equality follows from Lemma 4.3.

By Lemma 4.4, we can rewrite the last expression above as

$$\begin{aligned}
(x-1)^r p_{r,n}(x) - x^r p_{0,n}(x) &= \sum_{m=0}^{r-1}\left(\sum_{p=0}^{\min(m,n)}\left(\binom{r}{r-(m-p)}(-1)^{r-(m-p)} F_{n-p+1}^{(r)}\right)\right) x^m \\
&= (-1)^r \sum_{m=0}^{r-1}\left(\sum_{p=0}^{n}\left(\binom{r}{m-p}(-1)^{(m-p)} F_{n-p+1}^{(r)}\right)\right) x^m,
\end{aligned}$$

where we use the convention $\binom{a}{b} = 0$ for $b < 0$. Finally, we multiply through the equation above by $x^2 - x - 1$ and apply (4.2) to obtain the desired result. $\square$

We shall now apply Proposition 4.2 to count the number of real zeros of $p_{1,n}(x)$.

Theorem 4.5. *Let $n \in \mathbb{N} \cup \{0\}$. If n is even, then $p_{1,n}(x)$ has no real zeros. If n is odd, then $p_{1,n}(x)$ has a unique real zero, which lies in the interval $[-2, -\varphi)$.*

Proof. Substituting $r = 1$ into (4.3) and applying $F_n^{(r)} = F_{n-1}^{(r)} + F_n^{(r-1)}$ and the identities

$$F_n^{(1)} = F_{n+2} - 1, \tag{4.4}$$

$$F_{n+2}^{(1)} = F_{n+1}^{(1)} + F_n^{(1)} + 1, \quad n \geq 0, \tag{4.5}$$

we have

$$\begin{aligned} g_{1,n}(x) &= p_{1,n}(x)(x^2 - x - 1)(x - 1) \\ &= x^{n+3} - (F_{n+1}^{(1)} + F_n^{(1)} + 1)x^2 + F_n^{(1)}x + F_{n+1}^{(1)} \\ &= x^{n+3} - F_{n+2}^{(1)}x^2 + F_n^{(1)}x + F_{n+1}^{(1)}. \end{aligned} \tag{4.6}$$

Since $F_{n+1}^{(1)} \neq 0$, all roots of $g_{1,n}(x)$ must be nonzero. Assume first that n is even. Then

$$g_{1,n}(-x) = -x^{n+3} - (F_{n+2} + F_{n+1}^{(1)})x^2 - F_n^{(1)}x + F_{n+1}^{(1)}.$$

By Descartes' rule of signs, we have that $g_{1,n}(x)$ has at most two positive real zeros and exactly one negative real zero. Since $(x^2 - x - 1)(x - 1) = (x - \varphi)(x - \tau)(x - 1)$, where $\varphi = \frac{1+\sqrt{5}}{2} = 1.618\ldots > 0$ and $\tau = -1/\varphi < 0$, we have that $p_{1,n}(x)$ has no real zeros.

On the other hand, suppose that n is odd. Then

$$g_{1,n}(-x) = x^{n+3} - (F_{n+2} + F_{n+1}^{(1)})x^2 - F_n^{(1)}x + F_{n+1}^{(1)}.$$

Therefore, Descartes' rule of signs implies that $g_{1,n}(x)$ has exactly two positive real zeros and two negative real zeros. Hence $p_{1,n}(x)$ must have a unique (negative) real zero. Note that $p_{1,1}(x) = x + 2$ and $p_{1,3}(x) = x^3 + 2x^2 + 4x + 7$, whose real zeros are -2 and $-1.866\ldots$, respectively. We have $F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}$ (Cristea, Martinjak & Urbiha, 2016). Hence for any $l \geq 4$

$$\begin{aligned}
F_{l+1}^{(1)} &= \sum_{k=1}^{\lfloor \frac{l+2}{2} \rfloor} \binom{l-(k-2)}{k} \\
&= \binom{l+1}{1} + \binom{l}{2} + \sum_{k=3}^{\lfloor \frac{l+2}{2} \rfloor} \binom{l-(k-2)}{k} \\
&< \binom{l}{0} + \binom{l}{1} + \binom{l}{2} + \sum_{k=3}^{l} \binom{l}{k} = \sum_{k=0}^{l} \binom{l}{k} = 2^l.
\end{aligned}$$

Thus, substituting $x = -2$ into (4.6), we have that for $n \geq 5$

$$\begin{aligned}
g_{1,n}(-2) &= 2^{n+3} - 3F_{n+1}^{(1)} - 6F_n^{(1)} - 4 \\
&> 2^{n+3} - 3 \cdot 2^n - 6 \cdot 2^{n-1} - 2^2 \\
&> 2^{n+3} - 6 \cdot 2^n - 2^{n+1} = 0.
\end{aligned}$$

Next, we apply (4.4) and the Binet's formula $F_n = \frac{\varphi^n - \tau^n}{\sqrt{5}}$ in the evaluation of $g_{1,n}(x)$ at $x = -\varphi$ as follows:

$$\begin{aligned}
g_{1,n}(-\varphi) &= \varphi^{n+3} - F_{n+2}^{(1)}\varphi^2 - F_n^{(1)}\varphi + F_{n+1}^{(1)} \\
&= \varphi^{n+3} - \left(\frac{\varphi^{n+4} - \tau^{n+4}}{\sqrt{5}} - 1\right)\varphi^2 - \left(\frac{\varphi^{n+2} - \tau^{n+2}}{\sqrt{5}} - 1\right)\varphi + \left(\frac{\varphi^{n+3} - \tau^{n+3}}{\sqrt{5}} - 1\right) \\
&= \varphi^{n+3} + \varphi^2 + \varphi - 1 + \frac{\tau^{n+1}(-\tau^2 + \tau - 1) - \varphi^{n+6}}{\sqrt{5}} \\
&= \varphi^{n+3} + 2\varphi - \frac{2}{\sqrt{5}\varphi^{n+1}} - \frac{\varphi^{n+6}}{\sqrt{5}} \\
&= \varphi^{n+6}\left(\frac{1}{\varphi^3} + \frac{2}{\varphi^{n+5}} - \frac{2}{\sqrt{5}\varphi^{2n+7}} - \frac{1}{\sqrt{5}}\right),
\end{aligned}$$

where we have used the fact that φ and τ are roots of $x^2 - x - 1$ and $\varphi = -1/\tau$. Let

$$f(x) := \frac{1}{\varphi^3} + \frac{2}{\varphi^{x+5}} - \frac{2}{\sqrt{5}\varphi^{2x+7}} - \frac{1}{\sqrt{5}}.$$

Then by calculation we have $f(1) = -0.1114... < 0$ and $f'(x) = \frac{2\log\varphi}{\varphi^{x+5}}\left(\frac{2}{\sqrt{5}\varphi^{x+2}} - 1\right) < 0$ for every $x \geq 1$. As a consequence, $g_{1,n}(-\varphi) = \varphi^{n+6}f(n) < 0$ for every $n \geq 1$. By the intermediate value theorem, we can conclude that $g_{1,n}(x)$ has a real zero $x_0 \in$

$(-2, -\varphi)$. Since none of the zeros of $(x^2 - x - 1)(x - 1)$ is in this interval, x_0 must be the real zero of $p_{1,n}(x)$ and the proof is completed. $\square$

Remark 4.6. *It is worth noting that, for odd n, the sequence of real zeros of $p_{1,n}(x)$ seems to increase monotonically to $-\varphi$, according to our numerical data which are shown in Table 4.1. The fact that this sequence converges to $-\varphi$ follows directly from Theorem 4.11 below, yet we still cannot rigorously prove that it is monotone. This phenomenon looks almost identical to that in the case of $p_{0,n}(x)$, except that the sequence of their real zeros decreases monotonically from -1 to $-\varphi$ (Garth, Mill & Mitchell, 2007). We also hypothesize from numerical results that Theorem 4.5 is true for any $p_{r,n}(x)$.*

Table 4.1 Real zeros of $p_{1,n}(x)$ for some odd n

n	Real zero of $p_{1,n}(x)$
1	-2
11	$-1.7078955262722178552\ldots$
111	$-1.6273100491998780011\ldots$
1111	$-1.6189641192562181501\ldots$

Conjecture 4.7. *Let $r \geq 2$ and $n \in \mathbb{N} \cup \{0\}$. If n is even, then $p_{r,n}(x)$ has no real zeros. If n is odd, then $p_{r,n}(x)$ has a unique real zero, which lies in the interval $[-r-1, -\varphi)$. Moreover, the sequence of real zeros of $p_{r,n}(x)$, with n odd, increases monotonically to $-\varphi$.*

One should be able to prove the first part of Conjecture 4.7 for some small values of $r > 1$ with the aid of Proposition 4.2, Descartes' rule of signs, and some suitable manipulation. We give examples for the case $r = 2$ and $r = 3$ below. Locating the real zeros of $p_{r,n}(x)$ for arbitrary r is a much harder problem and it definitely requires much more delicate analysis.

Example 4.8. *By Proposition 4.2 with $r = 2$, we have*

$$\begin{aligned} g_{2,1}(x) &= p_{2,1}(x)(x^2 - x - 1)(x - 1)^2 \\ &= x^5 - 7x^3 + 7x^2 + 2x - 3, \quad \textit{and for} \quad n \geqslant 2, \\ g_{2,n}(x) &= p_{2,n}(x)(x^2 - x - 1)(x - 1)^2 \\ &= x^{n+4} - (3F_{n+1}^{(2)} - 2F_n^{(2)} - F_{n-1}^{(2)} + F_{n-2}^{(2)})x^3 + (2F_{n+1}^{(2)} + F_n^{(2)} - F_{n-1}^{(2)})x^2 \\ &\quad + (F_{n+1}^{(2)} - F_n^{(2)})x - F_{n+1}^{(2)}. \end{aligned}$$

Since $F_{n+1}^{(2)} - F_n^{(2)} > 0$ and $F_{n+1}^{(2)} - F_{n-1}^{(2)} > 0$, we have

$$\begin{aligned} 3F_{n+1}^{(2)} - 2F_n^{(2)} - F_{n-1}^{(2)} + F_{n-2}^{(2)} &= 2F_{n+1}^{(2)} - 2F_n^{(2)} + F_{n+1}^{(2)} - F_{n-1}^{(2)} + F_{n-2}^{(2)} \\ &= 2(F_{n+1}^{(2)} - F_n^{(2)}) + (F_{n+1}^{(2)} - F_{n-1}^{(2)}) + F_{n-2}^{(2)} > 0 \end{aligned}$$

Similarly, we have $2F_{n+1}^{(2)} + F_n^{(2)} - F_{n-1}^{(2)} > 0$ because $F_n^{(2)} - F_{n-1}^{(2)} > 0$.
Since $F_{n+1}^{(2)} \neq 0$, all roots of $g_{2,n}(x)$ must be nonzero. Assume first that n is even. Then

$$\begin{aligned} g_{2,n}(-x) = x^{n+4} &+ (3F_{n+1}^{(2)} - 2F_n^{(2)} - F_{n-1}^{(2)} + F_{n-2}^{(2)})x^3 + (2F_{n+1}^{(2)} + F_n^{(2)} - F_{n-1}^{(2)})x^2 \\ &- (F_{n+1}^{(2)} - F_n^{(2)})x - F_{n+1}^{(2)}. \end{aligned}$$

By Descartes' rule of signs, we have that $g_{2,n}(x)$ has at most three positive real zeros and exactly one negative real zero. Since $(x^2-x-1)(x-1)^2 = (x-\varphi)(x-\tau)(x-1)^2$, where $\varphi > 0$ and $\tau < 0$, we have that $p_{2,n}(x)$ has no real zeros.
On the other hand, suppose that n is odd. Then

$$\begin{aligned} g_{2,n}(-x) = -x^{n+4} &+ (3F_{n+1}^{(2)} - 2F_n^{(2)} - F_{n-1}^{(2)} + F_{n-2}^{(2)})x^3 + (2F_{n+1}^{(2)} + F_n^{(2)} - F_{n-1}^{(2)})x^2 \\ &- (F_{n+1}^{(2)} - F_n^{(2)})x - F_{n+1}^{(2)}. \end{aligned}$$

Therefore, Descartes' rule of signs implies that $g_{2,n}(x)$ has exactly three positive real zeros and two negative real zeros. Hence $p_{2,n}(x)$ must have a unique (negative) real zero. Note that $p_{2,1}(x) = x + 3$ and $p_{2,3}(x) = x^3 + 2x^2 + 4x + 7$, whose real zeros are -3 and $-2.4645\ldots$, respectively.

Example 4.9. *By Proposition 4.2 with $r = 3$, we have*

$$\begin{aligned}
g_{3,1}(x) &= p_{3,1}(x)(x^2-x-1)(x-1)^3\\
&= x^6 - 11x^4 + 19x^3 - 6x^2 - 7x + 4,\\
g_{3,2}(x) &= p_{3,2}(x)(x^2-x-1)(x-1)^3\\
&= x^7 - 25x^4 + 49x^3 - 18x^2 - 18x + 11, \quad \textit{and for} \quad n \geqslant 3,\\
g_{3,n}(x) &= p_{3,n}(x)(x^2-x-1)(x-1)^3\\
&= x^{n+5} - (4F_{n+1}^{(3)} - 5F_n^{(3)} + F_{n-1}^{(3)} + 2F_{n-2}^{(3)} - F_{n-3}^{(3)})x^4\\
&\quad + (5F_{n+1}^{(3)} - F_n^{(3)} - 2F_{n-1}^{(3)} + F_{n-2}^{(3)})x^3 - (F_{n+1}^{(3)} + 2F_n^{(3)} - F_{n-1}^{(3)})x^2\\
&\quad - (2F_{n+1}^{(3)} - F_n^{(3)})x + F_{n+1}^{(3)}.
\end{aligned}$$

Using Lemma 2.30, we have

$$\begin{aligned}
4F_{n+1}^{(3)} - 5F_n^{(3)} + F_{n-1}^{(3)} + 2F_{n-2}^{(3)} - F_{n-3}^{(3)} &= 4(F_{n+1}^{(3)} - F_n^{(3)}) - F_n^{(3)} + (F_{n-1}^{(3)} + F_{n-2}^{(3)})\\
&\quad + F_{n-2}^{(3)} - F_{n-3}^{(3)}\\
&= 4\left(F_{n-1}^{(3)} + \binom{n+2}{2}\right) - F_n^{(3)} + \left(F_n^{(3)} - \binom{n+1}{2}\right)\\
&\quad + F_{n-2}^{(3)} - F_{n-3}^{(3)}\\
&= 4F_{n-1}^{(3)} + 4\binom{n+2}{2} - \binom{n+1}{2} + F_{n-2}^{(3)} - F_{n-3}^{(3)}\\
&= 4F_{n-1}^{(3)} + 3\binom{n+2}{2} + \left(\binom{n+2}{2} - \binom{n+1}{2}\right)\\
&\quad + F_{n-2}^{(3)} - F_{n-3}^{(3)} > 0
\end{aligned}$$

because $\binom{n+2}{2} - \binom{n+1}{2} > 0$ and $F_{n-2}^{(3)} - F_{n-3}^{(3)} > 0$. Next, we have

$$\begin{aligned}
5F_{n+1}^{(3)} - F_n^{(3)} - 2F_{n-1}^{(3)} + F_{n-2}^{(3)} &= F_{n+1}^{(3)} - F_n^{(3)} + 4F_{n+1}^{(3)} - 2F_{n-1}^{(3)} + F_{n-2}^{(3)}\\
&= F_{n+1}^{(3)} - F_n^{(3)} + 2(2F_{n+1}^{(3)} - F_{n-1}^{(3)}) + F_{n-2}^{(3)} > 0
\end{aligned}$$

because $F_{n+1}^{(3)} - F_n^{(3)} > 0$ and $2F_{n+1}^{(3)} - F_{n-1}^{(3)} > 0$.

Finally, it is obvious that $F_{n+1}^{(3)} + 2F_n^{(3)} - F_{n-1}^{(3)} > 0$ and $2F_{n+1}^{(3)} - F_n^{(3)} > 0$.

Since $F_{n+1}^{(3)} \neq 0$, all roots of $g_{3,n}(x)$ must be nonzero. Assume first that n is even. Then

$$\begin{aligned} g_{3,n}(-x) = -x^{n+5} &- (4F_{n+1}^{(3)} - 5F_n^{(3)} + F_{n-1}^{(3)} + 2F_{n-2}^{(3)} - F_{n-3}^{(3)})x^4 \\ &- (5F_{n+1}^{(3)} - F_n^{(3)} - 2F_{n-1}^{(3)} + F_{n-2}^{(3)})x^3 - (F_{n+1}^{(3)} + 2F_n^{(3)} - F_{n-1}^{(3)})x^2 \\ &+ (2F_{n+1}^{(3)} - F_n^{(3)})x + F_{n+1}^{(3)}. \end{aligned}$$

By Descartes' rule of signs, we have that $g_{3,n}(x)$ has at most four positive real zeros and exactly one negative real zero. Since $(x^2 - x - 1)(x - 1)^3 = (x - \varphi)(x - \tau)(x - 1)^3$, where $\varphi > 0$ and $\tau < 0$, we have that $p_{3,n}(x)$ has no real zeros.
On the other hand, suppose that n is odd. Then

$$\begin{aligned} g_{3,n}(-x) = x^{n+5} &- (4F_{n+1}^{(3)} - 5F_n^{(3)} + F_{n-1}^{(3)} + 2F_{n-2}^{(3)} - F_{n-3}^{(3)})x^4 \\ &- (5F_{n+1}^{(3)} - F_n^{(3)} - 2F_{n-1}^{(3)} + F_{n-2}^{(3)})x^3 - (F_{n+1}^{(3)} + 2F_n^{(3)} - F_{n-1}^{(3)})x^2 \\ &+ (2F_{n+1}^{(3)} - F_n^{(3)})x + F_{n+1}^{(3)}. \end{aligned}$$

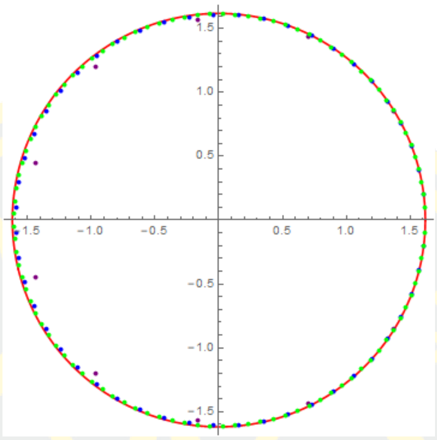
Therefore, Descartes' rule of signs implies that $g_{3,n}(x)$ has exactly four positive real zeros and two negative real zeros. Hence $p_{3,n}(x)$ must have a unique (negative) real zero. Note that $p_{3,1}(x) = x + 4$ and $p_{3,3}(x) = x^3 + 4x^2 + 11x + 25$, whose real zeros are -4 and $-3.0696\ldots$, respectively.

Next, we turn our attention to the complex zeros of $p_{r,n}(x)$. Let us first recall a known result about the zeros of $p_{0,n}(x)$, which is proven in (Garth et al., 2007).

Theorem 4.10. *(Garth et al., 2007) The zeros of $p_{0,n}(x)$ approach φ in modulus as $n \to \infty$.*

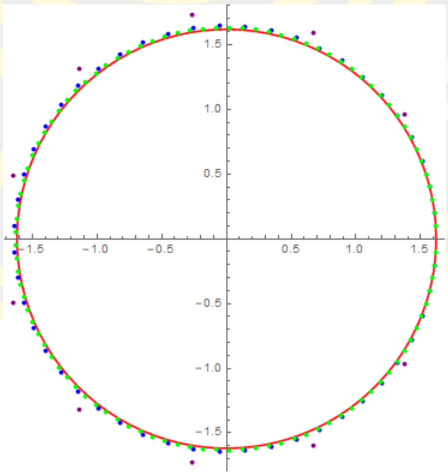
This theorem simply means the zeros of $p_{0,n}(x)$ get arbitrarily close to the circle $|z| = \varphi$ as n increases without bound, as visualized in Figure 4.1.

Figure 4.1 The roots of $p_{0,n}(x)$, with $n = 10$ (purple), 50 (blue), 100 (green) and the circle $|z| = \varphi$ (red)



One might be tempted to ask whether the zeros of $p_{r,n}(x)$ have similar behavior for $r \geq 1$. It turns out that this is true at least for $r = 1$. As opposed to $p_{0,n}(x)$, the zeros of $p_{1,n}(x)$ approach $|z| = \varphi$ from *outside* of the circle, as shown in Figure 4.2.

Figure 4.2 The roots of $p_{1,n}(x)$, with $n = 10$ (purple), 50 (blue), 100 (green) and the circle $|z| = \varphi$ (red)



Theorem 4.11. *The zeros of $p_{1,n}(x)$ approach φ in modulus as $n \to \infty$.*

Proof. Recall from (4.6) that

$$g_{1,n}(z) = (z^2 - z - 1)(z - 1)p_{1,n}(z) = z^{n+3} + F_n^{(1)} z + F_{n+1}^{(1)} - F_{n+2}^{(1)} z^2.$$

Assume first that $|z| = c$, where $1 < c < \varphi$, and let $P_n(z) = z^{n+3} + F_n^{(1)} z + F_{n+1}^{(1)}$ and $Q_n(z) = -F_{n+2}^{(1)} z^2$. Since $x^2 - x - 1 = (x - \varphi)(x - \tau)$ and the coefficients of $p_{1,n}(x)$ are non-negative, it is easily seen that $P_n(c) + Q_n(c) = g_{1,n}(c) < 0$. Using the triangle inequality, we have

$$|P_n(z)| \le P_n(c) < -Q_n(c) = |Q_n(z)|.$$

Hence $g_{1,n}(z) = P_n(z) + Q_n(z)$ has the same number of zeros interior to $|z| = c$ as does $Q_n(z)$ by Rouché's Theorem (Theorem2.24). Since $Q_n(z)$ has two (repeated) roots at $z = 0$ and 1 and τ are zeros of $g_{1,n}(z)$ inside the circle $|z| = c$, it follows that $p_{1,n}(z)$ has no zeros inside the circle $|z| = c$. Moreover, since $c < \varphi$ is arbitrary, we have that $p_{1,n}(z)$ has no zeros inside the circle $|z| = \varphi$.

Now suppose $|z| = c > \varphi$ and let $P_n(z) = -F_{n+2}^{(1)} z^2 + F_n^{(1)} z + F_{n+1}^{(1)}$ and $Q_n(z) = z^{n+3}$. By (4.4) and Binet's formula, we have

$$\begin{aligned} |P_n(z)| &\le F_{n+2}^{(1)} c^2 + F_n^{(1)} c + F_{n+1}^{(1)} \\ &< \left(\frac{\varphi^{n+4} - \tau^{n+4}}{\sqrt{5}}\right) c^2 + \left(\frac{\varphi^{n+2} - \tau^{n+2}}{\sqrt{5}}\right) c + \left(\frac{\varphi^{n+3} - \tau^{n+3}}{\sqrt{5}}\right). \end{aligned}$$

Since $c > \varphi > 1$ and $\tau = -1/\varphi$, we have that for any $l \ge 1$

$$\frac{1}{c^n}\left(\frac{\varphi^{n+l} - \tau^{n+l}}{\sqrt{5}}\right) \to 0 \text{ as } n \to \infty.$$

Therefore, for all sufficiently large n,

$$|P_n(z)| < c^{n+2} + c^{n+1} + c^n < c^{n+3} = |Q_n(z)|.$$

Then Rouché's Theorem implies that all zeros of $g_{1,n}(z) = P_n(z) + Q_n(z)$ are inside the circle $|z| = c$ for all sufficiently large n. Since the three zeros of $(z^2 - z - 1)(z - 1)$ are inside or on $|z| = \varphi$, all zeros of $p_{1,n}(z)$ are in the annulus $\varphi < |z| < c$ for large n. Since $c > \varphi$ is arbitrary, we have that the zeros of $p_{1,n}(z)$ approach $|z| = \varphi$ as $n \to \infty$. $\square$

We have shown in Theorem 4.5 that for odd n the real zeros of $p_{1,n}(x)$ is in the interval $[-2, -\varphi)$. Theorem 4.11 gives more information about convergence of the sequence of these zeros.

Corollary 4.12. *The sequence of real zeros of $p_{1,n}(x)$, with n odd, converges to $-\varphi$.*

Observing from graphs of zeros of $p_{r,n}(x)$ for different values of r and n (see Figures 4.3-4.5 below), we come up with the following conjecture.

Conjecture 4.13. *For $r \geq 2$, the zeros of $p_{r,n}(x)$ approach φ in modulus as $n \to \infty$.*

Figure 4.3 The roots of $p_{2,n}(x)$, with $n = 10$ (purple), 50 (blue), 100 (green) and the circle $|z| = \varphi$ (red)

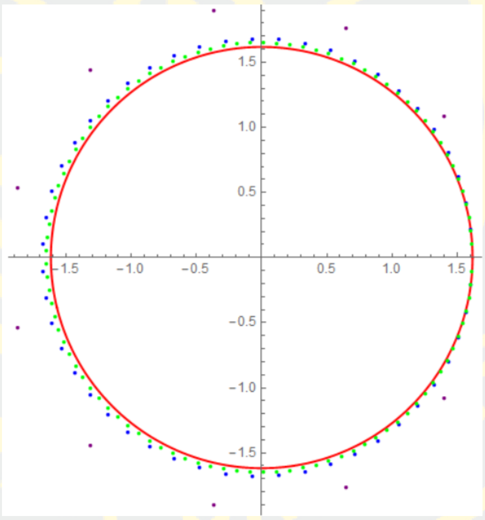


Figure 4.4 The roots of $p_{5,n}(x)$, with $n = 10$ (purple), 50 (blue), 100 (green) and the circle $|z| = \varphi$ (red)

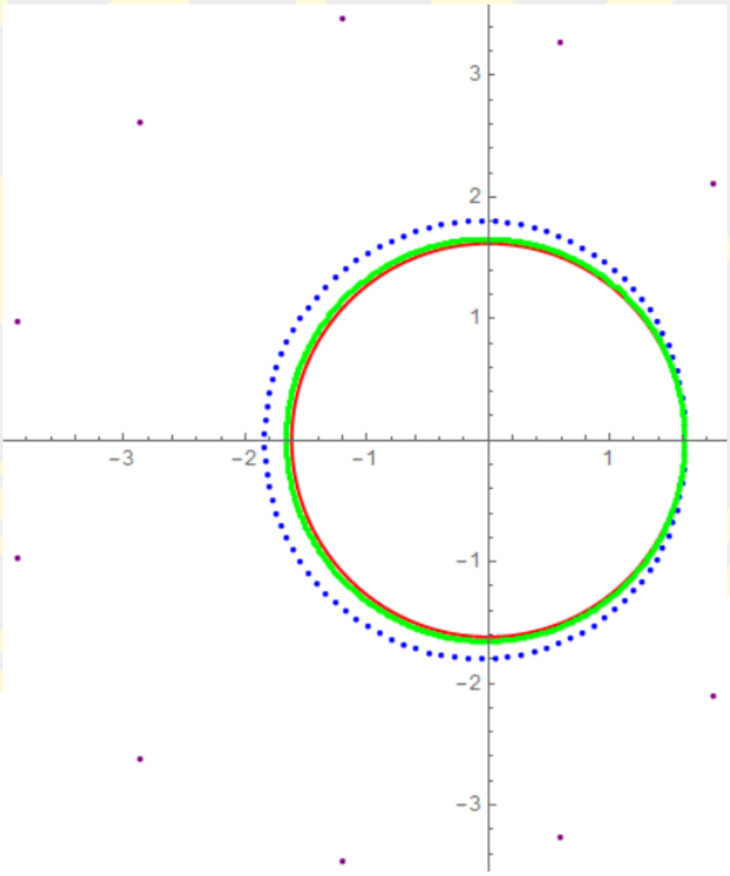
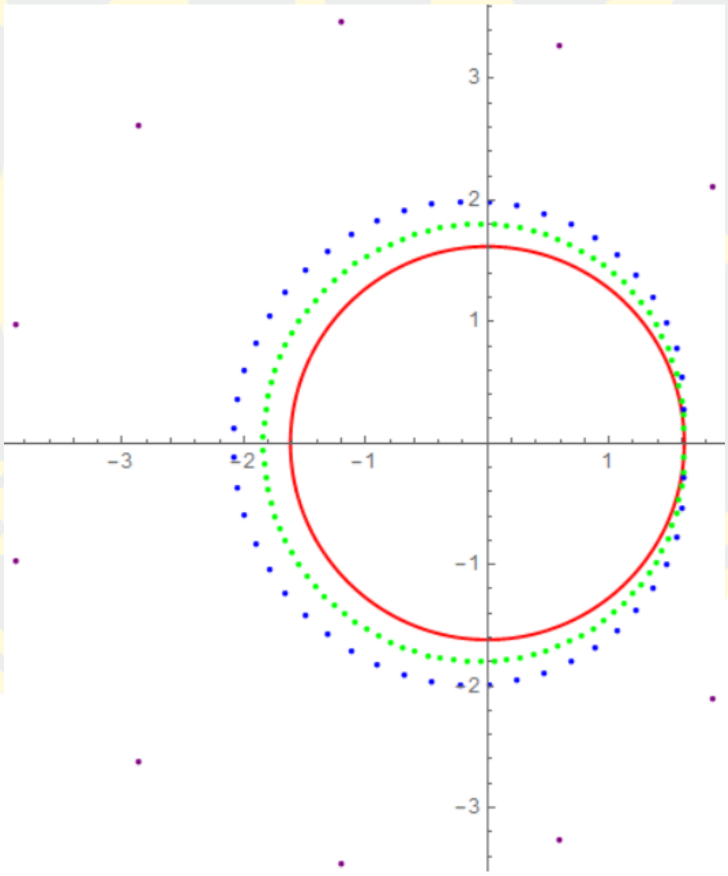


Figure 4.5 The roots of $p_{10,n}(x),$ with $n = 10$ (purple), 50 (blue), 100 (green) and the circle $|z| = \varphi$ (red)



4.2 Mahler measures of reduced-coefficient hFCPs

Following ideas in Lemma 2.2 and Lemma 2.3 of (Garth et al., 2007) we shall consider the Mahler measures of hFCPs whose coefficients are reduced modulo m with residue classes being restricted to the interval $[\lfloor -\frac{m}{2} \rfloor + 1, \lfloor \frac{m}{2} \rfloor]$. We start by proving the following result about periodicity of $\{F_n^{(1)}\}$ reduced modulo any positive integer m. For any $k \in \mathbb{Z}$ and $m \in \mathbb{N}$, we let $[k]_m$ denote the integer in $[\lfloor -\frac{m}{2} \rfloor + 1, \lfloor \frac{m}{2} \rfloor]$ which is congruent to k modulo m.

Theorem 4.14. *Let $m \in \mathbb{N}$. The sequence $\{[F_n^{(1)}]_m\}_{n=0}^{\infty}$ is periodic.*

Proof. Consider the set $\{([F_0^{(1)}]_m, [F_1^{(1)}]_m), ([F_1^{(1)}]_m, [F_2^{(1)}]_m), ([F_2^{(1)}]_m, [F_3^{(1)}]_m), \ldots\}$. Since there are totally m^2 distinct pairs of elements from the complete residue system modulo m, there exist $1 \leq k < l$ such that $([F_k^{(1)}]_m, [F_{k+1}^{(1)}]_m) = ([F_l^{(1)}]_m, [F_{l+1}^{(1)}]_m)$. Hence it follows from (4.5) that

$$[F_{k-1}^{(1)}]_m \equiv [F_{k+1}^{(1)}]_m - [F_k^{(1)}]_m - 1 \equiv [F_{l+1}^{(1)}]_m - [F_l^{(1)}]_m - 1 \equiv [F_{l-1}^{(1)}]_m \pmod{m},$$
$$[F_{k+2}^{(1)}]_m \equiv [F_{k+1}^{(1)}]_m + [F_k^{(1)}]_m + 1 \equiv [F_{l+1}^{(1)}]_m + [F_l^{(1)}]_m + 1 \equiv [F_{l+2}^{(1)}]_m \pmod{m},$$

implying $([F_{k-1}^{(1)}]_m, [F_k^{(1)}]_m) = ([F_{l-1}^{(1)}]_m, [F_l^{(1)}]_m)$ and $([F_{k+1}^{(1)}]_m, [F_{k+2}^{(1)}]_m) = ([F_{l+1}^{(1)}]_m, [F_{l+2}^{(1)}]_m)$. One can apply the same argument inductively to see that $([F_0^{(1)}]_m, [F_1^{(1)}]_m) = ([F_p^{(1)}]_m, [F_{p+1}^{(1)}]_m)$ for some $p \geq 1$ and $[F_{hp+j}^{(1)}]_m = [F_j^{(1)}]_m$ for every $h \geq 1$ and $0 \leq j \leq p-1$. $\square$

If p is the smallest positive integer for which $([F_p^{(1)}]_m, [F_{p+1}^{(1)}]_m) = ([F_0^{(1)}]_m, [F_1^{(1)}]_m)$, then we call $[F_0^{(1)}]_m, [F_1^{(1)}]_m, \ldots, [F_{p-1}^{(1)}]_m$ the *period* of the sequence $\{[F_n^{(1)}]_m\}_{n=0}^{\infty}$. Garth et al. showed that the Fibonacci sequence reduced modulo a Lucas number L_n has two (simple) types of period, depending on the parity of n (Garth et al., 2017). It can be shown that this is also the case for the hyperfibonacci sequence of the first generation.

Proposition 4.15. *If $n \geq 3$ is odd, then the period of the hyperfibonacci sequence of the first generation reduced modulo L_n, with residue classes adjusted to range between $\lfloor -\frac{L_n}{2} \rfloor + 1$ and $\lfloor \frac{L_n}{2} \rfloor$, is*

$$F_2 - 1, F_3 - 1, \ldots, F_n - 1, -F_{n-1} - 1, F_{n-2} - 1, \ldots, F_1 - 1, -F_0 - 1, 0.$$

If $n \geq 8$ is even, then the period of the hyperfibonacci sequence of the first generation reduced modulo L_n, with residue classes adjusted as above, is

$$\begin{aligned}&F_2 - 1, F_3 - 1, \ldots, F_n - 1, -F_{n-1} - 1, F_{n-2} - 1, \ldots, F_2 - 1, -F_1 - 1,\\ &-(F_0 + 1), -(F_1 + 1), \ldots, -(F_n + 1), F_{n-1} - 1, -F_{n-2} - 1, \ldots, -F_0 - 1, 0\end{aligned}$$

Proof. Let $n \geq 3$ be an odd integer. Using the well-known identity $L_n = F_{n+1} + F_{n-1}$, we have

$$\begin{aligned}L_n = F_{n+1} + F_{n-1} &= F_n + F_{n-1} + F_{n-1}\\ &> F_n + F_{n-1} + F_{n-2} = 2F_n,\end{aligned}$$

implying $0 \leq F_l - 1 \leq F_n - 1 < F_n < \frac{L_n}{2}$ for any $2 \leq l \leq n$. By (4.4), we immediately have that the first $n-1$ terms in the period are $F_2 - 1, F_3 - 1, \ldots, F_n - 1$. The next n terms in the period are obtained from (4.4) and the identity

$$F_{n+k} = F_k L_n + (-1)^k F_{n-k}, \quad 1 \leq k \leq n,$$

which is proven in Lemma 2.2 of (Garth et al., 2017). Since $0 \leq F_j < F_n < \frac{L_n}{2}$ for $0 \leq j \leq n-1$, we have that $\lfloor -\frac{L_n}{2} \rfloor + 1 \leq (-1)^k F_{n-k} - 1 \leq \lfloor \frac{L_n}{2} \rfloor$ for $1 \leq k \leq n$. Notice that $F_1 - 1 = 0$ and $-F_0 - 1 = -1$, so the next term in the period is 0 by (4.5). By the same argument, the next two terms in the sequence reduced modulo L_n become 0 and 1, which repeat the initial terms of the period.

The case when $n \geq 8$ is even can be verified in a similar manner by splitting the first $4n$ terms of the sequence into $n-1$, $n-1$, $n+1$, and $n+1$ terms, respectively, and applying the following identities

$$\begin{aligned}F_{2n+k} &= F_{n+k} L_n - F_k,\\ F_{3n+k} &= (F_{2n+k} - F_k) L_n + (-1)^{k+1} F_{n-k}, \quad 1 \leq k \leq n,\end{aligned}$$

which are given in Lemma 2.3 of (Garth et al., 2017). □

Remark 4.16. *For $n = 2, 4,$ and 6, there are minor discrepancies between the periods of $\{F_k^{(1)}\}_{k=0}^{\infty}$ reduced modulo L_n and those in Proposition 4.14 for even $n \geq 8$. Their periods can be written explicitly as follows:*

$n = 2 : 0, 1, -1, 1, 1, 0, -1, 0,$

$n = 4 : 0, 1, 2, -3, 0, -2, -1, -2, -2, -3, 3, 1, -2, 0, -1, 0,$

$n = 6 : 0, 1, 2, 4, 7, -6, 2, -3, 0, -2, -1, -2, -2, -3, -4, -6, 9, 4, -4, 1, -2, 0, -1, 0.$

For $m \in \mathbb{N}$, we define $p_{r,n}^{(m)}(x)$ as the polynomial $p_{r,n}(x)$ whose coefficients are reduced modulo m with residue classes adjusted to be in the interval $[\lfloor -\frac{m}{2} \rfloor + 1, \lfloor \frac{m}{2} \rfloor]$. In the next theorem, we obtain a result about Mahler measures of $p_{1,n}^{(m)}(x)$, which is identical to Theorem 4.1 in (Garth et al., 2017) and can be proven using the same arguments.

Proposition 4.17. *Let $m \geq 2$ and let t be the length of the period of the hyperfibonacci sequence of the first generation reduced modulo m. Then*

$$M\left(p_{1,nt-1}^{(m)}(x)\right) = M\left(p_{1,nt-2}^{(m)}(x)\right) = M\left(p_{1,t-2}^{(m)}(x)\right)$$

for every $n \in \mathbb{N}$.

Proof. Let $a_1, \ldots, a_t$ be the period of the hyperfibonacci sequence of the first generation reduced modulo m. Then we have

$$p_{1,t-2}^{(m)}(x) = x^{t-2} + a_3 x^{t-3} + \cdots + a_{t-1} x + a_t,$$

since $a_1 = 0$ and $a_2 = 1$. Let $C_j(x) = x^j + x^{j-1} + \cdots + x + 1$, where $j \geqslant 1$. We will show that for every $n \geqslant 1$

$$p_{1,nt-2}^{(m)}(x) = p_{1,t-2}^{(m)}(x) \cdot C_{n-1}(x^t).$$

For $n = 1$, the above equation is obvious.

Let $k \in \mathbb{N}$ and suppose $p^{(m)}_{1,kt-2}(x) = p^{(m)}_{1,t-2}(x) \cdot C_{k-1}(x^t)$. Then

$$\begin{aligned}
p^{(m)}_{1,kt+t-2}(x) &= x^{kt+t-2} + a_3 x^{kt+t-3} + a_4 x^{kt+t-4} + \cdots + a_{t-2}x^{kt+2} + a_{t-1}x^{kt+1} + a_t x^{kt} \\
&\quad + a_1 x^{kt-1} + a_2 x^{kt-2} + a_3 x^{kt-3} + \cdots + a_{t-1}x^{kt-t+1} + a_t x^{kt-t} \\
&\quad + a_1 x^{(k-1)t-1} + a_2 x^{(k-1)t-2} + \cdots + a_{t-1}x + a_t \\
&= x^{kt+t-2} + a_3 x^{kt+t-3} + a_4 x^{kt+t-4} + \cdots + a_{t-2}x^{kt+2} + a_{t-1}x^{kt+1} + a_t x^{kt} \\
&\quad + p^{(m)}_{1,kt-2}(x) \\
&= x^{kt}\left(x^{t-2} + a_3 x^{t-3} + a_4 x^{t-4} + \cdots + a_{t-2}x^2 + a_{t-1}x + a_t\right) + p^{(m)}_{1,kt-2}(x) \\
&= x^{kt}\left(p^{(m)}_{1,t-2}(x)\right) + p^{(m)}_{1,kt-2}(x) \\
&= x^{kt}\left(p^{(m)}_{1,t-2}(x)\right) + C_{k-1}(x^t) \cdot p^{(m)}_{1,t-2}(x) \\
&= \left(x^{kt} + C_{k-1}(x^t)\right) \cdot p^{(m)}_{1,t-2}(x) \\
&= \left(x^{kt} + x^{(k-1)t} + x^{(k-2)t} + \cdots + x^t + 1\right) \cdot p^{(m)}_{1,t-2}(x) \\
&= C_k(x^t) \cdot p^{(m)}_{1,t-2}(x).
\end{aligned}$$

Similarly, we have that for every $n \in \mathbb{N}$

$$p^{(m)}_{1,nt-1}(x) = x p^{(m)}_{1,t-2}(x) C_{n-1}(x^t).$$

Since Mahler measure is multiplicative and the zeros of $C_{n-1}(x^t)$ are roots of unity, the desired result follows immediately. □

The final result given in (Garth et al., 2017) is the following theorem:

Theorem 4.18. *For $n, l \in \mathbb{N}$, we have*

$$M\left(p^{(L_n)}_{0,2ln-2}(x)\right) = M\left(p^{(L_n)}_{0,2ln-1}(x)\right) = \varphi^{n-1}.$$

Indeed, what Garth et al. proved is the following identity:

$$M\left(p^{(L_n)}_{0,2n-2}(x)\right) = \varphi^{n-1} \quad \text{for every } n \geq 1, \tag{4.7}$$

which results from Lemma 2.2 and Lemma 2.3 of (Garth et al., 2007) and some tricky manipulation. The extended version in Theorem 4.18 is then just a consequence of Theorem 4.1 of (Garth, Mills & Mitchell, 2007). One might expect to see a similar result for $p_{r,n}^{(m)}(x)$ with $r \geq 1$. Unfortunately, this does not seem to be the case, even for $r = 1$. The coefficients of $p_{0,k}^{(L_n)}(x)$ are, up to sign, Fibonacci numbers, while those of $p_{1,k}^{(L_n)}(x)$ have an extra term -1 by Proposition 4.15, which makes a significant difference for Mahler measures. Note that (4.7) is equivalent to

$$\frac{M\left(p_{0,2n}^{(L_{n+1})}\right)}{M\left(p_{0,2n-2}^{(L_n)}\right)} = \varphi \quad \text{for every } n \geq 1.$$

If we replace 0 with $r \geq 1$, the quotient on the left-hand side is apparently not constant as n varies. However, we propose here an interesting conjecture based on our numerical results.

Conjecture 4.19. *Let $r \in \mathbb{N}$. The following convergence is valid:*

$$\lim_{n\to\infty} \frac{M\left(p_{r,2n}^{(L_{n+1})}\right)}{M\left(p_{r,2n-2}^{(L_n)}\right)} = \varphi.$$

CHAPTER 5

CONCLUSION AND DISCUSSION

In this chapter, we summarize the results about hFCPs that we obtain in this thesis. First, we study various properties of hyper-Fibonacci numbers and hFCPs, as stated in the first research objective. We prove that if n is even, then $p_{1,n}(x)$ has no real zeros and if n is odd, then $p_{1,n}(x)$ has a unique real zero which lies in the interval $[-2, -\varphi)$. This result partially answers the problem stated in the second research objective. Furthermore, we show that the complex zeros of $p_{1,n}(x)$ approach φ in modulus as $n \to \infty$. This result explains the behavior of moduli of zeros of some hFCP's $p_{r,n}(x)$ for some r and large n and satisfies the third research objective. We also proved that for $m \in \mathbb{N}$ the sequence $\{[F_n^{(1)}]_m\}_{n=0}^{\infty}$ obtained from $\{F_n^{(1)}\}_{n=0}^{\infty}$ using reduction modulo m is periodic. This result satisfies our last research objective. Then we show that the sequence $\{F_k^{(1)}\}_{k=1}^{\infty}$ reduced modulo m is periodic and it has simple periods when $m = L_n$, as shown in Proposition 4.15. Finally, we consider Mahler measure of $p_{1,n}^{(m)}(x)$, which is obtained from $p_{1,n}(x)$ by reducing the coefficients modulo m. We prove that for any $m, n \in \mathbb{N}$ with $m \geqslant 2$

$$M\left(p_{1,nt-1}^{(m)}(x)\right) = M\left(p_{1,nt-2}^{(m)}(x)\right) = M\left(p_{1,t-2}^{(m)}(x)\right),$$

where t is the number of terms in one period of the hyper-Fibonacci sequence reduced modulo m.

5.1 Suggestions

Although we succeed in proving many results about $p_{1,n}(x)$, several problems remain open, especially for $p_{r,n}(x)$ with $r > 1$. These problems are stated in Conjecture 4.7, Conjecture 4.13 and Conjecture 4.19. We firmly believe that it is not out of reach to prove these conjectures for some small $r > 1$. For example, choosing $r = 2$ and 3, we are able to prove using Proposition 4.2 that $p_{r,n}(x)$ has no real zeros if n is even and $p_{r,n}(x)$ has a unique real zero if n is odd. That said, we are still unable to find an easy way to prove our results in full generality.

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